Decision Making for Inconsistent Expert Judgments Using Signed Probabilities

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Why probabilities?

- Most ways to think *rationally* lead to probability measures a la Kolmogorov:
  - Pascal (motivated by Antoine Gombaud, Chevalier de Méré).
  - Cox, Jaynes, Ramsey, de Finneti.
  - Venn, von Mises.
- Originally, probabilities were meant to be normative, and not descriptive.

Contextuality and the logic of QM

- QM observable operators do not fit into a standard boolean algebra (quantum lattice).
- Such lattice leads to nonmonotonic upper probability measures or to signed probabilities.[1]
- Upper probabilities are consequence of strong contextual (inconsistent) correlations.

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Contextuality and the logic of QM

- QM observable operators do not fit into a standard boolean algebra (quantum lattice).
- Such lattice leads to nonmonotonic upper probability measures or to signed probabilities.[1]
- Upper probabilities are consequence of strong contextual (inconsistent) correlations.
- How to think "rationally" about inconsistencies?
  - Quantum descriptions?
  - Nonstandard (negative) probabilities?

[1] .

Outline

1. Inconsistent Beliefs
2. Modeling Inconsistent Beliefs
   - Bayesian Model
   - Quantum Model
   - Signed Probability Model
3. Final remarks

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Inconsistencies

- In logic, any two or more sentences are inconsistent if it is possible to derive from them a contradiction, i.e., if there exists an A such that $(A \wedge \neg A)$ is a theorem.[2]
- If a set of sentences is inconsistent, then it is trivial.
- To see this, let's start with $A \wedge \neg A$ as true. Then A is also true. But since A is true, then so is $A \vee B$ for any B. But since $\neg A$ is true, it follows from conjunction elimination that B is necessarily true.
- Paraconsistent logics may be used to deal with inconsistent sentences without exploding.[3]

2 .

3 .

With probabilities

- Take **X**, **Y**, and **Z** as ± 1-valued random variables.
- The above example is equivalent to the deterministic case where

$$E(\mathbf{XY}) = E(\mathbf{XZ}) = E(\mathbf{YZ}) = -1.$$

- Clearly the correlations are too strong to allow for a joint probability distribution.

A subtler case

- Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be ± 1 random variables with zero expectation representing future trends on stocks of companies X, Y, and Z going up or down.
- Three experts, Alice, Bob, and Carlos, have beliefs about the relative behavior of pairs of stocks.
- There is no joint[4] for $E_A(\mathbf{XY}) = -1$, $E_B(\mathbf{XZ}) = -1/2$, $E_C(\mathbf{YZ}) = 0$, as

$$
\begin{aligned}
-1 \leq \quad & E(\mathbf{XY}) + E(\mathbf{XZ}) + E(\mathbf{YZ}) \leq \\
& 1 + 2\min\{E(\mathbf{XY}), E(\mathbf{XZ}), E(\mathbf{YZ})\}.
\end{aligned}
$$

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How to deal with inconsistencies?

- Question: what is the triple moment $E(\mathbf{XYZ})$?
- There are several approaches in the literature.
  - Paraconsistent logics.
  - Consensus reaching.
  - Bayesian.
- Here we will examine two possible alternatives:
  - Quantum.
  - Signed probabilities.

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Bayesian Model: Priors

- We start with Alice, Bob, and Carlos as experts, and Deanna Troy as a decision maker.
- In the Bayesian approach, Deanna starts with a prior probability distribution.
- If we assume she knows nothing about X, Y, and Z, it is reasonable that she sets

$$p^D_{xyz} = p^D_{\bar{x}yz} = \cdots = p^D_{\overline{xyz}} = \frac{1}{16}.$$

Model of experts

- In order to apply Bayes's theorem, Deanna needs to have a model of the experts (likelihood function).
- Imagine that an oracle tells Deanna that tomorrow the actual correlation $E(\mathbf{XY}) = -1$.
- If Deanna thinks her expert is good, knowing that $E(\mathbf{XY}) = -1$ means that she should think that $p_{xy\cdot}$ and $p_{\overline{xy}\cdot}$ should be highly improbable for Alice, whereas $p_{\overline{x}y\cdot}$ and $p_{x\overline{y}\cdot}$ highly probable.
- For instance, Deanna might propose that the likelihood function is given by

$$p_{xy\cdot} = p_{\overline{xy}\cdot} = 1 - \frac{1}{4}(1-\epsilon_A)^2,$$

$$p_{\overline{x}y\cdot} = p_{\overline{x}y\cdot} = -\frac{1}{4}(1-\epsilon_A)^2,$$

where $E_A(\mathbf{XY}) = \epsilon_A$.
- Similarly for Bob and Carlos.

Applying Bayes's Theorem

- Deanna can use Bayes's theorem to revise her prior belief's about X, Y, and Z.
- For example,

$$p_{xyz}^{D|A} = k\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8},$$

where

$$\begin{aligned} k^{-1} &= \left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}+\left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}+\left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} \\ &+\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}+\left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}+\left[\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8} \\ &+\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}+\left[1-\frac{1}{4}\left(1-\epsilon_A\right)^2\right]\frac{1}{8}, \\ &= \frac{1}{2}. \end{aligned}$$

Incorporating Bob and Carlos's opinion

- Deanna can now revise her posterior $p_{xyz}^{D|A}$ using once again Bayes's theorem.
- She gets

$$p_{xyz}^{D|AB} = \frac{1}{32}\left[\left(\epsilon_A^2 - 2\epsilon_A - 3\right)\epsilon_B^2 + \left(-2\epsilon_A^2 + 4\epsilon_A + 6\right)\epsilon_B - 3\epsilon_A^2 + 6\epsilon_A + 9\right].$$

- A third application of the theorem gives us $p_{xyz}^{D|ABC}$.
- Similar computations can be carried out for the other atoms.

Example

- If $\epsilon_A = 0$, $\epsilon_B = -\frac{1}{2}$, $\epsilon_C = -1$, we have

$$p^{D|ABC}_{xyz} = p^{D|ABC}_{x\overline{y}z} = p^{D|ABC}_{\overline{x}y\overline{z}} = p^{D|ABC}_{\overline{xyz}} = 0,$$

$$p^{D|ABC}_{\overline{x}yz} = p^{D|ABC}_{x\overline{yz}} = \frac{7}{68},$$

and

$$p^{D|ABC}_{xy\overline{z}} = p^{D|ABC}_{\overline{xy}z} = \frac{27}{68}.$$

- From the joint, we obtain, e.g.,

$$E\left(\mathbf{XYZ}\right) = 0.$$

Summary: Bayesian

- The Bayesian approach is the standard probabilistic approach for decision making.
- It is extremely sensitive on the prior distribution.
- Depends on the model of experts (likelihood function).
- Allows to compute a proper joint probability distribution.

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Quantum model

Theorem

Let $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ be three observables in a Hilbert space $\mathcal{H}$ with eigenvalues ± 1, and let them pairwise commute, and let the ± 1-valued random variable $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$ *represent the outcomes of possible experiments performed on a quantum system $|\psi\rangle \in \mathcal{H}$. Then, there exists a joint probability distribution consistent with all the possible outcomes of* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$.

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- Bell: "The only thing proved by impossibility proofs is the author's lack of imagination."

Inserting different contexts: measurement

- If we want to model the above correlations, we need to explicitly include the context.
- E.g.

$$E_A(\mathbf{XY}) = \langle\psi_{xy}|\hat{X}\hat{Y}|\psi_{xy}\rangle,$$

where $|\psi\rangle_{xy} \neq |\psi\rangle_{yz} \neq |\psi\rangle_{xz}$.

- For instance, consider the three orthonormal states $|A\rangle$, $|B\rangle$, and $|C\rangle$, and let

$$|\psi\rangle = c_{xy}|\psi_{xy}\rangle \otimes |A\rangle + c_{xz}|\psi_{xz}\rangle \otimes |B\rangle + c_{yz}|\psi_{yz}\rangle \otimes |C\rangle.$$

- We can compute a joint, and therefore $E(\mathbf{XYZ})$, from $|\psi\rangle$.
- There are infinite number of $|\psi\rangle$ satisfying the correlations, and $-1 \leq E(\mathbf{XYZ}) \leq 1$.

Summary: quantum

- Provides a way to compute the triple moment from a context-dependent vector.
- Imposes no constraint on the relative weights or triple moment.
- Doesn't tell us what is our best bet.

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Kolmogorov model

- Kolmogorov axiomatized probability in a set-theoretic way, with the following simple axioms.

$$\text{A1.}\ \ 1 \geq P(A) \geq 0$$

$$\text{A2.}\ \ P(\Omega) = 1$$

$$\text{A3.}\ \ P(A \cup B) = P(A) + P(B)$$

Upper and lower probabilities

- How do we deal with inconsistencies?
- de Finetti: relax Kolmogorov's axiom A2:

$$P^{*}\left(A\cup B\right)\geq P^{*}\left(A\right)+P^{*}\left(B\right)$$

or

$$P_{*}\left(A\cup B\right)\leq P_{*}\left(A\right)+P_{*}\left(B\right).$$

- Subjective meaning: bounds of best measures for inconsistent beliefs (imprecise probabilities).

Upper and lower probabilities

- Consequence:

$$M^* = \sum_i P_i^* > 1,$$

$$M_* = \sum_i P_{*i} < 1.$$

- M^* and M_* should be as close to one as possible.
- Inequalities and nonmonotonicity make it hard to compute upper and lowers for practical problems.

Workaround?

- Define $M^T = \sum_i |p(A_i)|$.
- Instead of violating A2, relax A1:

$$\text{A'1. } p_i \text{ are such that } M^T \text{ is minimum.}$$

$$\text{A'2. } p(A_i \cup A_j) = p(A_i) + p(A_j), \; i \neq j,$$

$$\text{A'3. } \sum_i p(A_i) = 1.$$

- A_i (probability of atom i)[5] can now be negative.
- p defines an optimal upper probability distribution by simply setting all negative probability atoms to zero.
- Atoms with negative probability are thought subjectively as impossible events.

[5] The definition of atoms might be difficult once we relax A1, but for finite probability spaces this is not a problem.

Why negative probabilities?

- We can compute them easily (compared to uppers/lowers).
- May be helpful to think about certain contextual problems (e.g. non-signaling conditions, counterfactual reasoning).
- They have a meaning in terms of subjective probability.

Marginals from Alice, Bob, and Carlos

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} + p_{xy\bar{z}} + p_{x\overline{yz}} + p_{\bar{x}y\bar{z}} + p_{\overline{xy}z} + p_{\overline{xyz}} = 1, \tag{1}$$

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\overline{yz}} - p_{\bar{x}y\bar{z}} - p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{2}$$

$$p_{xyz} + p_{\bar{x}yz} - p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\overline{yz}} + p_{\bar{x}y\bar{z}} - p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{3}$$

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} - p_{xy\bar{z}} - p_{x\overline{yz}} - p_{\bar{x}y\bar{z}} + p_{\overline{xy}z} - p_{\overline{xyz}} = 0, \tag{4}$$

$$p_{xyz} - p_{\bar{x}yz} - p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\overline{yz}} - p_{\bar{x}y\bar{z}} + p_{\overline{xy}z} + p_{\overline{xyz}} = 0, \tag{5}$$

$$p_{xyz} - p_{\bar{x}yz} + p_{x\bar{y}z} - p_{xy\bar{z}} - p_{x\overline{yz}} + p_{\bar{x}y\bar{z}} - p_{\overline{xy}z} + p_{\overline{xyz}} = -\frac{1}{2}, \tag{6}$$

$$p_{xyz} + p_{\bar{x}yz} - p_{x\bar{y}z} - p_{xy\bar{z}} + p_{x\overline{yz}} - p_{\bar{x}y\bar{z}} - p_{\overline{xy}z} + p_{\overline{xyz}} = -1, \tag{7}$$

Signed Probabilities

$$
\begin{aligned}
p_{xyz} &= -p_{\overline{x}yz} &= -\frac{1}{8} - \delta, \\
p_{x\overline{y}z} &= p_{\overline{x}y\overline{z}} = & \frac{3}{16}, \\
p_{xy\overline{z}} &= p_{\overline{xy}z} = & \frac{5}{16}, \\
p_{x\overline{yz}} &= -p_{\overline{xyz}} = & -\delta,
\end{aligned}
$$

$$
E(\mathbf{XYZ}) = -\frac{1}{4} - 4\delta.
$$

- From A'1, we have as constraint

$$
-\frac{1}{8} \leq \delta \leq 0, \text{ which implies } -\tfrac{1}{4} \leq E\left(\mathbf{XYZ}\right) \leq \tfrac{1}{2}.
$$

Summary: signed probabilities

- Signed probabilities have a natural interpretation in terms of (subjective) upper probabilities.
- Minimization of M^{-} requires the improper distributions to approach as best as possible the rational proper jpd.
- This has a normative constraint on the choices of triple moment.

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Summary

- Standard Bayesian approach is sensitive to choices of prior and likelihood function (well-known problem).
  - "It ain't what you don't know that gets you into trouble. It's what you know for sure that just ain't so." -Mark Twain
  - E.g. say that Deanna starts with $E(\mathbf{XYZ}) = \epsilon$ as her prior. The posterior will give $E(\mathbf{XYZ}) = \epsilon$ regardless of Alice, Bob, and Carlos's opinions.
- The quantum-like approach, using vectors on a Hilbert space, seems to be too permissive, and to not have normative power. (Is it the only quantum model for it?)
  - Can we find some additional principle in QM to help with this?
- Negative probabilities, with the minimization of the negative mass, offers a lower and upper bound for values of triple moment.