Quantum Formalism Outside of Physics

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Weird quantum

- Quantum mechanics is strange:
  - No clear interpretation
  - **Apparently** contradictory statements (inconsistent histories)
- But is well described mathematically (Hilbert spaces or rigged Hilbert spaces)
  - Quantum probabilities

Cognitive models

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- Typical problems are perception, decision making, and learning.
- Cognitive models are mathematical models of the above problems using constraints from behavioral and cognitive sciences.
- Such models provide a more refined way to talk about features of certain cognitive processes.

Irrational behavior

Behavior violates Kolmogorov's axioms.

- Most cognitive models use standard probabilities satisfying Kolmogorov's axioms:
  - (i) $P(A) \geq 0$, $A \in \mathscr{F}$,
  - (ii) $P(\Omega) = 1$,
  - (iii) $P(A \cup B) = P(A) + P(B)$ for $A \cap B = \emptyset$.
- This is the case with Markov chain or Bayesian models, for example.

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  - (iii) $P(A \cup B) = P(A) + P(B)$ for $A \cap B = \emptyset$.
- This is the case with Markov chain or Bayesian models, for example.
- However, people do not behave according to Kolmogorov's axioms (we'll see examples later, as well as what this means).

Quantum models

- Since classical probability theory seems to be violated by human behavior, cognitive scientists looked for alternatives to it.
    - Extended probabilities
    - upper and lower probabilities
    - negative probabilities
    - quantum probabilities.
- Can we use of the mathematical apparatus of quantum mechanics to model human behavior?
    - This comes from the well-known fact that quantum mechanics violate Kolmogorov's axioms.

In a nutshell

- Cognitive modeling, mainly probabilistic ones, helps us understand quantitatively the brain.
- However, brain processes, or at least behavior, seem to violate classical probability theory.
- Can a quantum mechanical mathematical formalism help us model such processes?

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- However, brain processes, or at least behavior, seem to violate classical probability theory.
- Can a quantum mechanical mathematical formalism help us model such processes?
- To emphasize: we are not talking about quanta and mind here!

About the lectures

Structure.

Day 1: Basic issues on quantum mechanics (with emphases aspects relevant for cognition)

Day 2: Quantum in psychology

Day 3: Quantum in psychology, economics, and (perhaps) political sciences

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- Pushing quanta to outside of physics may help clarify what makes quanta different.
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"Religion without science is blind; science without religion is lame" - Albert Einstein

Metaphysics without physics is blind; physics without metaphysics is lame.

Outline

1. What is non-classical about quantum mechanics?
   - Non-determinism
   - Nonlocality
2. Contextuality, quantum, and probabilities
3. Quantum Cognition
   - Order Effect
   - The conjunction paradox and the two-slit
   - The prisoner's dilemma
   - A neurophysiological model for quantum cognition.
   - Beyond quantum cognition.

What makes QM different from CM?

- Non-determinism.
- Contextuality.
- Non-locality.

Outline

They are both deterministic

- Classical particle physics relies on Newton's equations of motion:

$$m\frac{\mathrm{d}^2\mathbf{r}\left(t\right)}{\mathrm{d}t^2}=\mathbf{F}\left(\mathbf{r},\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t},t\right).$$

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$$m\frac{\mathrm{d}^2\mathbf{r}(t)}{\mathrm{d}t^2} = \mathbf{F}\left(\mathbf{r}, \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, t\right).$$
- Quantum mechanics relies on Schroedinger's equation, e.g.
$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t).$$

Both are deterministic, but quantum is not

At least quantum measurement seems not to be

- But $\psi(\mathbf{r},t)$ in

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t)+V(\mathbf{r},t)\psi(\mathbf{r},t).$$

  only tells us the probability density, $p(\mathbf{r},t)=|\psi(\mathbf{r},t)|^2$, **if we measure its position**.

- For a general vector $|\psi\rangle$ and a projector P, the outcome is

$$c\hat{P}|\psi\rangle$$

  with probability $p=\left|\hat{P}|\psi\rangle\right|^2$.

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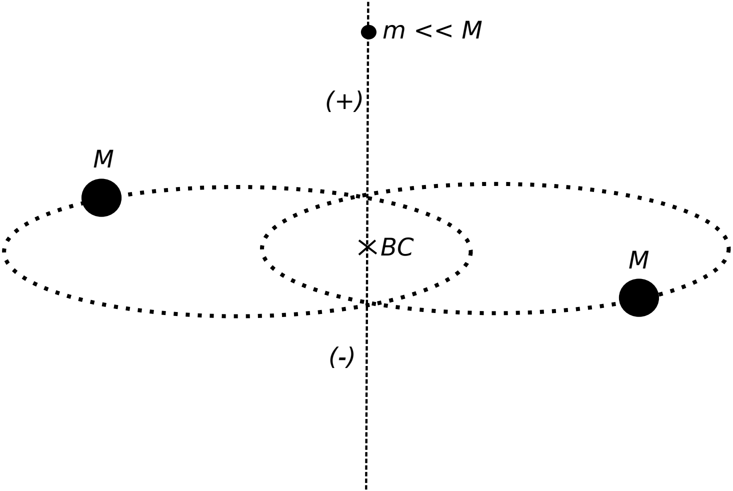
- A quantum measurement is apparently probabilistic (nondeterministic).
- Quantum non-determinism was recognized early on by

But do we care about determinism?

- Let us look at some classical (and therefore deterministic) examples:
  - Three-body problem.
  - Sinai billiard.

Three bodies under gravity.



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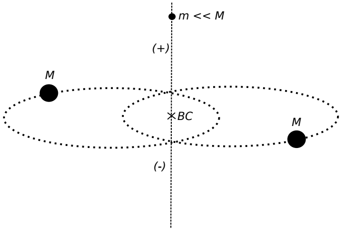

- A symbolic trajectory for m is a set of measurements of its position every time interval Δt: $S = (+, +, -, -, -, +, \ldots)$.

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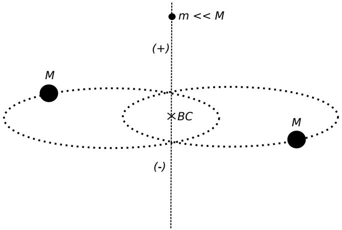
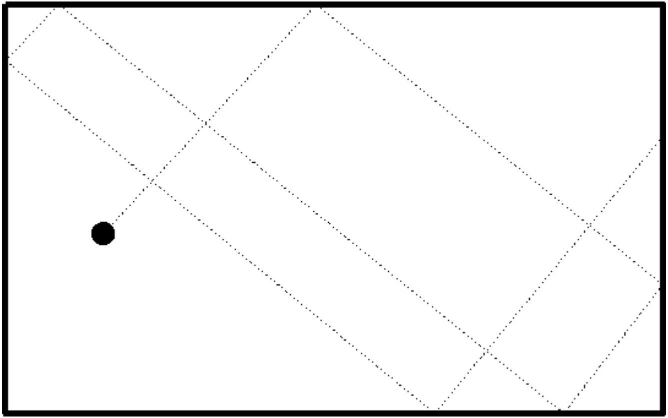
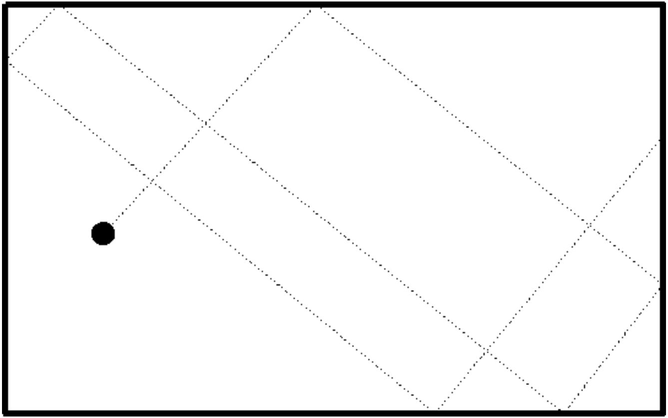


- A symbolic trajectory for m is a set of measurements of its position every time interval Δt: $S = (+,+,-,-,-,+,\ldots)$.
- Alekseev: If $m \ll M$, and if Δt is large enough, then S is isomorphic to a coin toss.

Billiard.

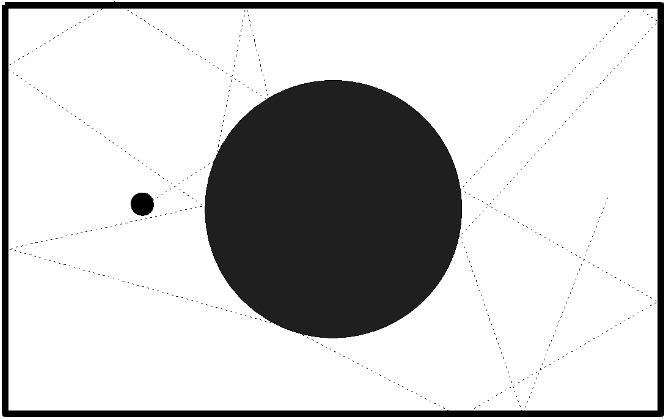


Billiard.



- Periodic trajectories.

Sinai's Billiard.



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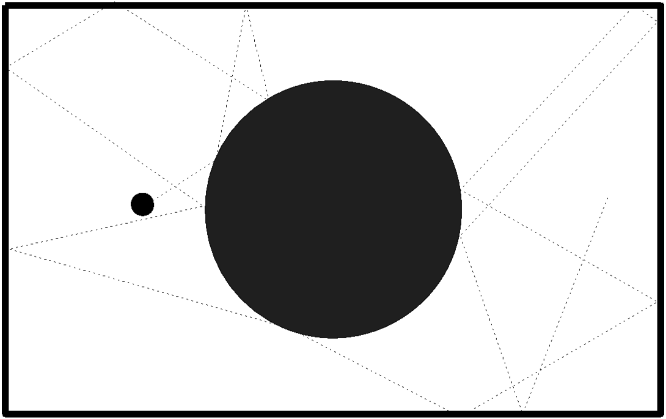
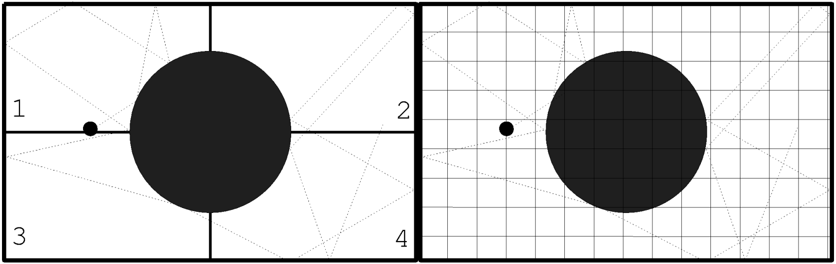


- Ergodic.

Ornstein's partition.



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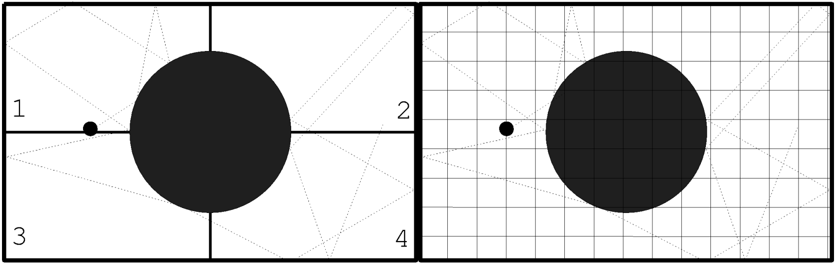

- Deterministic and probabilistic systems are ε-congruent.

Summarizing.

- Determinism does not imply predictability.
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- Determinism does not imply predictability.
- Deterministic complex systems behave in ways that are observationally equivalent to probabilistic systems.
- When observing a system that seems non-deterministic, it is possible we're observing a deterministic system with complex dynamics.
- Distinction between determinism and predictability was not known to founders of QM.

Outline

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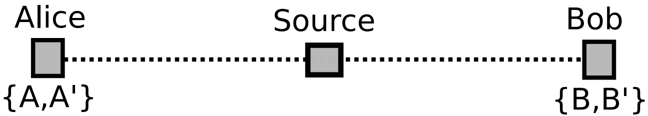
Bell-EPR experiment



- Only sixteen possibilities:
  - $A = 1, A' = 1, B = 1, B' = 1$
  - $A = 1, A' = 1, B = 1, B' = -1$
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  - $A = 1, A' = 1, B = -1, B' = -1$
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  - $\vdots$
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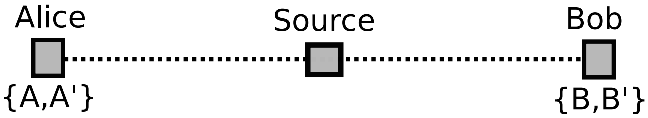

- Define $S = AB + AB' + A'B - A'B'$:
  - $A = 1, A' = 1, B = 1, B' = 1 \longrightarrow S = AB + AB' + A'B - A'B' = 2$
  - $A = 1, A' = 1, B = 1, B' = -1 \longrightarrow S = 2$
  - $A = 1, A' = 1, B = -1, B' = 1 \longrightarrow S = -2$
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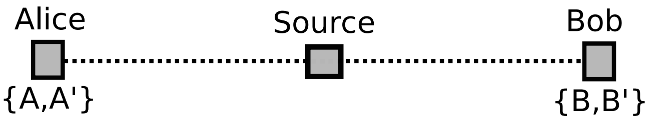
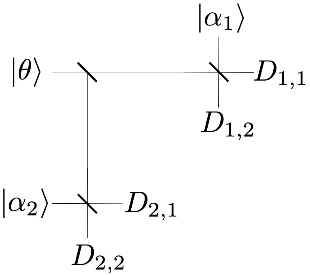


- Only sixteen possibilities.
- Define $S = AB + AB' + A'B - A'B'$
- Since for each possibility, $-2 \leq S \leq 2$, it follows that $-2 \leq \langle S \rangle \leq 2$.
- Quantum mechanics violates this inequality.
  - This is equivalent to the non-existence of a common cause that explains the correlations (non-locality)

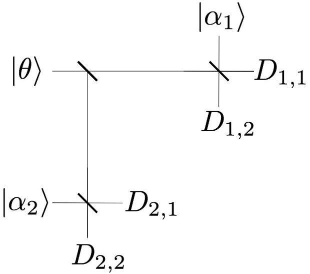
Homodyne detection



$$I_{1,1} - I_{1,2} \equiv V_1 = \frac{1}{4}\alpha\beta\cos(\theta - \theta_1)$$

$$I_{2,1} - I_{2,2} \equiv V_2 = \frac{1}{4}\alpha\beta\cos(\theta - \theta_2)$$

Interference effects



$$\rho(V_1, V_2) = \frac{\mathrm{Cov}(V_1, V_2)}{\sqrt{\mathrm{Var}(V_1)\mathrm{Var}(V_2)}}$$

$$\rho(V_1, V_2) = -\sin(\alpha_1 - \alpha_2).$$

- This correlation violates Bell's inequalities.
- However, proper violation has a thermodynamical cost.

Quantum and Classical Dynamics

- Classical:

$$\frac{dp}{dt} = \{p, H\}, \quad (1)$$

$$\frac{dq}{dt} = \{q, H\}, \quad (2)$$

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$$\frac{dp}{dt} = \{p, H\}, \quad (1)$$

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- Quantum:

$$\frac{d\hat{P}}{dt} = i\hbar\left[\hat{P}, \hat{H}\right],$$

$$\frac{d\hat{Q}}{dt} = i\hbar\left[\hat{Q}, \hat{H}\right].$$

- To preserve the algebra, we impose $[\hat{Q}, \hat{P}] = i\hbar$.

So what if $\hat{P}$ and $\hat{Q}$ don't commute?

- We saw that $[\hat{Q},\hat{P}]|\psi\rangle = \left(\hat{Q}\hat{P}-\hat{P}\hat{Q}\right)|\psi\rangle = i\hbar|\psi\rangle$.
- Suppose $|\psi\rangle$ is simultaneously an eigenstate of $\hat{P}$ and $\hat{Q}$.
- $\left(\hat{Q}\hat{P}-\hat{P}\hat{Q}\right)|\psi\rangle = \left(\hat{Q}p_0-\hat{P}q_0\right)|\psi\rangle = (q_0p_0-p_0q_0)|\psi\rangle = 0$. But this contradicts above!

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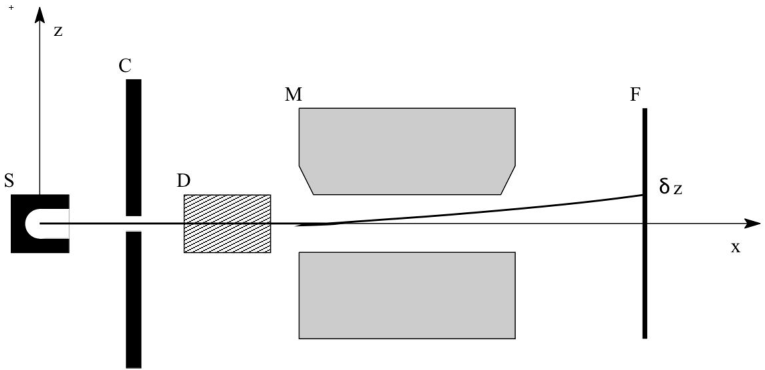
- If two observables do not commute, they are not simultaneously measurable.
- Complimentary observables are not simultaneously measurable.

Example: measuring the magnetic moment of an electron



Big problem?

- Classically, a measurement reveals the value of the quantity being measured.
  Let $\boldsymbol{\mu}$ be the a random variable representing spin. The experiment is measuring $\mu_z = \boldsymbol{\mu} \cdot \hat{\mathbf{z}} = \pm 1$ (in units where $\hbar = h/2\pi = 1$).

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- But our choice of direction is arbitrary for the experiment! So, it must also be true for any other directions that its spin component is either 1 or -1.
- Let us chose two new directions, $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ such that $\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 + \hat{\mathbf{z}} = 0$.
  It follows that $\boldsymbol{\mu} \cdot (\hat{\mathbf{x}}_1 + \hat{\mathbf{x}}_2 + \hat{\mathbf{z}}) = \pm 1 \pm 1 \pm 1 = 0$, a contradiction!

What is going on?

- There are a assumptions for the contradiction:
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- Spin cannot be independent of measurement.
- To avoid the inconsistency, we need to assume that $\boldsymbol{\mu}$ depends on the choice of measurement direction.
  The value of $\boldsymbol{\mu}$ depends on the experiment, i.e. it changes with the "context".
- Contextuality is at the heart of Heisenberg's uncertainty principle: the measuring of a quantity affects the state of a system.

Contextuality.

- Founders of QM were puzzled by the fundamental nature of non-commutativity.
- Bohr created (or borrowed) the concept of complementarity.
- We saw that this was the essence of contextuality: we cannot assign values to spin before measure, as measurement itself affects it.
- But contextuality can also be classical (we will see examples later).

Why probabilities?

- Let $\{P_i\}$ be a collection of propositions.
- How do we express a *rational* belief about such propositions if we are not sure about them?
  - Divisibility and comparability: the belief in a proposition is represented by a real number; this belief depends on what we know about this proposition
  - Logicality: beliefs should vary sensibly with the assessment of plausibilities, i.e. it should be consistent with logic (if P_1 is believed to be true, then $\neg P_1$ should be believed to be false).
  - Consistency: If a belief about a proposition can have different derivations, the outcomes must all be the same.

Cox's theorem

- If the rationality assumptions are true, then belief can be measured with a "belief" function p (probability) with the following properties:
  1. $1 \geq p(P_1) \geq 0$
  2. $p(A \cup B) = p(A) + p(B)$, if $A \cap B = \emptyset$.

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  2. $p(A \cup B) = p(A) + p(B)$, if $A \cap B = \emptyset$.
- Cox's theorem is (finite) equivalent to Kolmogorov's axioms.
  - $(\Omega, \mathscr{F}, p)$ with $p : \mathscr{F} \to [0,1]$ and

$$p(A \cup B) = p(A) + p(B), \text{ for } A, B \in \mathscr{F} \text{ and } A \cap B = \emptyset$$

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Properties

- A very powerful tool in probability theory are random variables.
- A random variable $\mathbf{R}:\Omega\rightarrow O$ is a measurable function in $(\Omega,\mathscr{F},p)$ from the sample space into a (measurable) set of outcomes.
- Simple examples:
    - $\Omega=\{h,t\}$, $p(h)=p(t)=1/2$; $\mathbf{R}:\Omega\rightarrow\{0,1\}$
    - $A=\{1,2,3,4,5,6\}$, $\Omega=A\times A$, $\mathbf{R}:\Omega\rightarrow\{2,3,4,\ldots,12\}$, $\mathbf{R}((a_1,a_2))=a_1+a_2$, where $(a_1,a_2)\in\Omega$.
- Random variables can be used to describe properties or experimental outcomes.
    - Their stochastic properties should match the properties of the experiment.

Contextuality comes from linguistics

- Consider $P =$ "Aristotle knew very little philosophy."

  Alice: P is true.

  Bob: P is false.

A different example

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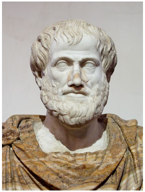
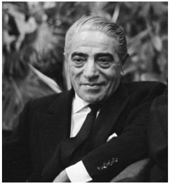
 $\neq$ 

What is contextuality in physics?

- The most famous example of contextuality in physics was given by Kochen and Specker.
  - For a Hilbert space of dimension 3:
    - a set of projection operators, P_i, corresponding to true or false propositions about the physical system
    - sets of contexts where *some* of those P_i's are compatible
    - no context-independent truth-values can be assigned to the outcomes of such measurements in all contexts
- In other words, we cannot assign a truth value to a P_i that is the same in one context as in another; the property P_i depends on the context.

A non-physical example of KS-type contextuality

- Let Alice, Carol, and Bob be three undergrads.
  - They are very competitive: whenever they are in the same room, they *always* disagree.
  - They are also very unpredictable: you never know whether they will repeat an answer to the same question.
- Let:
  - P_1 be the proposition "Alice answered *yes* to question Q."
  - P_2 be the proposition "Bob answered *yes* to question Q."
  - P_3 be the proposition "Carol answered *yes* to question Q."

A non-physical example of KS-type contextuality

- Let Alice, Carol, and Bob be three undergrads.
  - They are very competitive and unpredictable
- The corresponding r.v. for P_1, P_2, and P_3 are two-valued variables, either 1 (true) or 0 (false).
- Daniel goes to three classes, and observes $C_1 = (P_1, P_2)$, $C_2 = (P_1, P_3)$, and $C_3 = (P_2, P_3)$.
  - Let the following be true for each context:

$$\mathbf{P}_1 + \mathbf{P}_2 = 1,$$
$$\mathbf{P}_1 + \mathbf{P}_3 = 1,$$
$$\mathbf{P}_2 + \mathbf{P}_3 = 1.$$

- However, we can see that there is an inconsistency: even on the left, odd on the right.

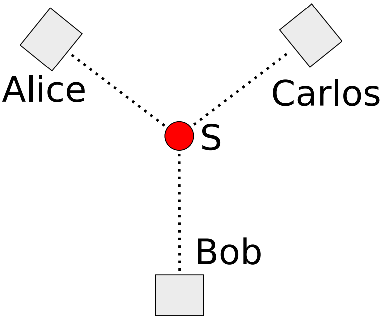
Inconsistency comes from assuming non-contextuality

- Contradiction comes from assuming that truth-values are the same in each context.
  - Say $\mathbf{P}_1 = 1$ and $\mathbf{P}_2 = 0$ in the context C_1, and $\mathbf{P}_1 = 1$ and $\mathbf{P}_3 = 0$ in context C_2.
  - But we know that in C_3 either $\mathbf{P}_2 = 0$ and $\mathbf{P}_3 = 1$ or $\mathbf{P}_2 = 1$ and $\mathbf{P}_3 = 0$, i.e. one of them is not the same as in the other contexts.
- That is why we say that the random variables in this example are contextual: they cannot be the same.

Kochen-Specker

$$
\begin{aligned}
\mathbf{v}_{0,0,0,1}+\mathbf{v}_{0,0,1,0}+\mathbf{v}_{1,1,0,0}+\mathbf{v}_{1,-1,0,0}&=1,\\
\mathbf{v}_{0,0,0,1}+\mathbf{v}_{0,1,0,0}+\mathbf{v}_{1,0,1,0}+\mathbf{v}_{1,0,-1,0}&=1,\\
\mathbf{v}_{1,-1,1,-1}+\mathbf{v}_{1,-1,-1,1}+\mathbf{v}_{1,1,0,0}+\mathbf{v}_{0,0,1,1}&=1,\\
\mathbf{v}_{1,-1,1,-1}+\mathbf{v}_{1,1,1,1}+\mathbf{v}_{1,0,-1,0}+\mathbf{v}_{0,1,0,-1}&=1,\\
\mathbf{v}_{0,0,1,0}+\mathbf{v}_{0,1,0,0}+\mathbf{v}_{1,0,0,1}+\mathbf{v}_{1,0,0,-1}&=1,\\
\mathbf{v}_{1,-1,-1,1}+\mathbf{v}_{1,1,1,1}+\mathbf{v}_{1,0,0,-1}+\mathbf{v}_{0,1,-1,0}&=1,\\
\mathbf{v}_{1,1,-1,1}+\mathbf{v}_{1,1,1,-1}+\mathbf{v}_{1,-1,0,0}+\mathbf{v}_{0,0,1,1}&=1,\\
\mathbf{v}_{1,1,-1,1}+\mathbf{v}_{-1,1,1,1}+\mathbf{v}_{1,0,1,0}+\mathbf{v}_{0,1,0,-1}&=1,\\
\mathbf{v}_{1,1,1,-1}+\mathbf{v}_{-1,1,1,1}+\mathbf{v}_{1,0,0,1}+\mathbf{v}_{0,1,-1,0}&=1.
\end{aligned}
$$

Another example: GHZ



Entangled three-particle state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+++\rangle - |---\rangle\right)$$

This state is the eigenvector of many different operators. E.g.

$$\begin{aligned}
\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|\psi\rangle &= \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|+++\rangle - \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|---\rangle \\
&= \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}(i)|++-\rangle - \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}(-i)|--+\rangle \\
&= \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}(i)(i)|+--\rangle - \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}(-i)(-i)|-++\rangle \\
&= \frac{1}{\sqrt{2}}(i)(i)|---\rangle - \frac{1}{\sqrt{2}}(-i)(-i)|+++\rangle \\
&= -\frac{1}{\sqrt{2}}|---\rangle + \frac{1}{\sqrt{2}}|+++\rangle = |\psi\rangle
\end{aligned}$$

Entangled three-particle state.

In addition to

$$\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|\psi\rangle = |\psi\rangle,$$

above, we have

$$\hat{\sigma}_{y,A}\hat{\sigma}_{x,B}\hat{\sigma}_{y,C}|\psi\rangle = |\psi\rangle,$$

$$\hat{\sigma}_{y,A}\hat{\sigma}_{y,B}\hat{\sigma}_{x,C}|\psi\rangle = |\psi\rangle.$$

But also

$$\hat{\sigma}_{x,A}\hat{\sigma}_{x,B}\hat{\sigma}_{x,C}|\psi\rangle = \frac{1}{\sqrt{2}}\hat{\sigma}_{x,A}\hat{\sigma}_{x,B}\hat{\sigma}_{x,C}\left(|+++\rangle - |---\rangle\right)$$
$$= -\frac{1}{\sqrt{2}}|+++\rangle + \frac{1}{\sqrt{2}}|---\rangle = -|\psi\rangle.$$

What does it mean?

$$\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|\psi\rangle = |\psi\rangle.$$

$$\mathbf{X}_A(\omega)\mathbf{Y}_B(\omega)\mathbf{Y}_C(\omega) = 1$$

For the examples shown.

$$\hat{\sigma}_{x,A}\hat{\sigma}_{y,B}\hat{\sigma}_{y,C}|\psi\rangle = |\psi\rangle, \qquad \mathbf{X}_A(\omega)\mathbf{Y}_B(\omega)\mathbf{Y}_C(\omega) = 1$$

$$\hat{\sigma}_{y,A}\hat{\sigma}_{x,B}\hat{\sigma}_{y,C}|\psi\rangle = |\psi\rangle, \qquad \mathbf{Y}_A(\omega)\mathbf{X}_B(\omega)\mathbf{Y}_C(\omega) = 1$$

$$\hat{\sigma}_{y,A}\hat{\sigma}_{y,B}\hat{\sigma}_{x,C}|\psi\rangle = |\psi\rangle, \qquad \mathbf{Y}_A(\omega)\mathbf{Y}_B(\omega)\mathbf{X}_C(\omega) = 1,$$

$$\hat{\sigma}_{x,A}\hat{\sigma}_{x,B}\hat{\sigma}_{x,C}|\psi\rangle = -|\psi\rangle, \qquad \mathbf{X}_A(\omega)\mathbf{X}_B(\omega)\mathbf{X}_C(\omega) = -1.$$

Again, big problem!

$$\mathbf{X}_A\mathbf{Y}_B\mathbf{Y}_C = 1$$
$$\mathbf{Y}_A\mathbf{X}_B\mathbf{Y}_C = 1$$
$$\mathbf{Y}_A\mathbf{Y}_B\mathbf{X}_C = 1$$

$$(\mathbf{X}_A\mathbf{Y}_B\mathbf{Y}_C)(\mathbf{Y}_A\mathbf{X}_B\mathbf{Y}_C)(\mathbf{Y}_A\mathbf{Y}_B\mathbf{X}_C) = 1.$$
$$(\mathbf{X}_A\mathbf{X}_B\mathbf{X}_C)\left(\mathbf{Y}_A^2\mathbf{Y}_B^2\mathbf{Y}_C^2\right) = 1.$$
$$\mathbf{X}_A\mathbf{X}_B\mathbf{X}_C = 1.$$

Again, big problem!

$$\mathbf{X}_A \mathbf{Y}_B \mathbf{Y}_C = 1$$
$$\mathbf{Y}_A \mathbf{X}_B \mathbf{Y}_C = 1$$
$$\mathbf{Y}_A \mathbf{Y}_B \mathbf{X}_C = 1$$

$$(\mathbf{X}_A \mathbf{Y}_B \mathbf{Y}_C)(\mathbf{Y}_A \mathbf{X}_B \mathbf{Y}_C)(\mathbf{Y}_A \mathbf{Y}_B \mathbf{X}_C) = 1.$$
$$(\mathbf{X}_A \mathbf{X}_B \mathbf{X}_C)\left(\mathbf{Y}_A^2 \mathbf{Y}_B^2 \mathbf{Y}_C^2\right) = 1.$$
$$\mathbf{X}_A \mathbf{X}_B \mathbf{X}_C = 1.$$

But before

$$\mathbf{X}_A \mathbf{X}_B \mathbf{X}_C = -1!$$

How is it possible?

- The assumption that $\mathbf{Y}_A^2 = 1$ is based on the idea that the random variable is the same in both experiments (i.e., there is a joint probability space for both experimental conditions).
- But $\mathbf{Y}_A$, the outcome of a spin measurement by Alice, must depend on the choice of experimental setup, i.e., whether Bob and Carlos decide to measure $\hat{\mathbf{x}}$-spin or $\hat{\mathbf{y}}$-spin.
- But all experimenters, Alice, Bob, or Carlos, may decide at the last second whether they want to measure $\hat{\mathbf{x}}$-spin or $\hat{\mathbf{y}}$-spin.

Extension to non-perfect correlation

- Cases above were for perfect correlations: not a realistic assumption.
- For imperfect correlations, we need to use probabilities.
- But for non-contextual variables, logical entailments lead to probabilities satisfying certain inequalities.

An example with imperfect correlation

- Suppes-Zanotti inequality

$$-1 \leq \langle \mathbf{AB} \rangle + \langle \mathbf{AC} \rangle + \langle \mathbf{BC} \rangle \leq 1 + 2\min\{\langle \mathbf{AB} \rangle, \langle \mathbf{AC} \rangle, \langle \mathbf{BC} \rangle\},$$

where **A**, **B**, and **C** are ± 1-valued random variables.

- To see that this must be the case, we can examine all logical possibilities for each product:

$$(\mathbf{AB} = 1 \& \mathbf{AC} = 1) \rightarrow \mathbf{BC} = 1$$
$$(\mathbf{AB} = 1 \& \mathbf{AC} = -1) \rightarrow \mathbf{BC} = -1$$
$$(\mathbf{AB} = -1 \& \mathbf{AC} = 1) \rightarrow \mathbf{BC} = -1$$
$$(\mathbf{AB} = -1 \& \mathbf{AC} = -1) \rightarrow \mathbf{BC} = 1.$$

- Since each line above add to numbers that are either -1 or 3, their convex combination must be greater than -1.

Imperfect correlation for GHZ

- For the GHZ example above, we have A, B, C, and $D = ABC$ as our simplified set of random variables.
- Let us examine the following logical possibilities:

$$(A = 1 \& B = 1 \& C = 1) \rightarrow D = 1$$
$$(A = 1 \& B = 1 \& C = -1) \rightarrow D = -1$$
$$\vdots$$
$$(A = -1 \& B = -1 \& C = -1) \rightarrow D = -1.$$

- A convex sum of all those possibilities imply that

$$-2 \leq E(A) + E(B) + E(C) - E(D) \leq 2$$

(and permutations of the $-$ sign).
- Those are necessary and sufficient conditions for existence of a joint.

Contextuallity in social sciences

- Common:
  - semantics and pragmatics
  - order effect in psychology
  - perception

Contextuallity in social sciences

- Common:
  - semantics and pragmatics
  - order effect in psychology
  - perception
- Does quantum bring something new?

Outline

What is order effect?

- It is well known in social sciences that order matter:
  - Two questions, A and B, cannot be asked simultaneously
  - We need to choose order: AB or BA?
  - Order matters

Example of order effect: Clinton/Gore

- Consider the two questions:
  - A: Do you think Clinton is honest and trustworthy?
  - B: Do you think Gore is honest and trustworthy?
- Context 1: A first, then B in the context of A (Gore is in the comparative context)
  - A: 53% yes; B: 65% yes.
- Context 2: B first, then A in the context of B (Clinton is in the comparative context)
  - A: 59% yes; B: 76% yes

A quantum model for order effect

- From Busemeyer and Wang:

Postulate 1: A person's belief about an object in question is represented by a state vector in a multidimensional feature space (a vector space)

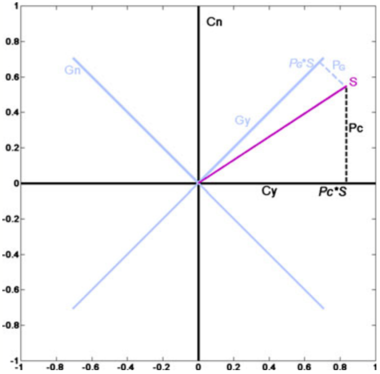
Postulate 2: A potential response to a question is represented by a subspace of the multidimensional feature space.

Postulate 3: The probability of responding to an opinion question equals the squared length of the projection of the state vector onto the response subspace.

Postulate 4: The updated belief state after deciding an answer to a question equals the normalized projection on the subspace representing the answer.

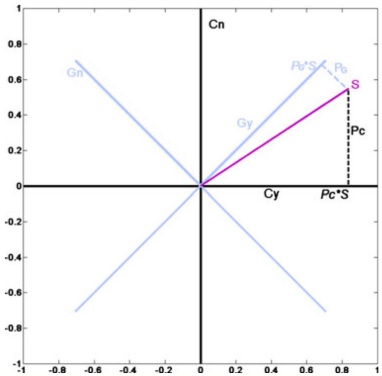
Quantum model for Clinton/Gore

Feature space of 2-dim
S is the mental state
C_y and C_n are basis for Clinton
G_y and G_n are basis for Gore



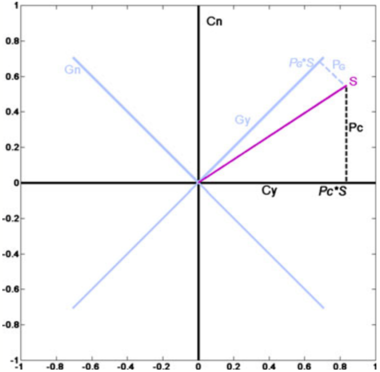
Quantum model for Clinton/Gore

C_y and C_n are basis for Clinton
G_y and G_n are basis for Gore
To satisfy the empirical data, we can have S (the respondent's belief concerning "whether Clinton is honest and trustworthy") as $(.8367, .5477)$ in the Clinton Basis.
S in the Gore basis (respondent's belief about "whether Gore is honest and trustworthy") is $(.9789, .2043)$.



Quantum model for Clinton/Gore

C_y and C_n are basis for Clinton
G_y and G_n are basis for Gore
$S = (.8367, .5477)$ in the Clinton Basis.
If we first ask Clinton, $p(C_y) = .7$
If we first ask Gore, $p(G_y) = .96$.
In the comparative context, $p(C_y) = p(G_y) = .5$



The QQ equality

- The example above showed order effect
- But what does this mean quantum formalism is a good description of this?
- QQ (quantum question order model) equality is an interesting result.
- $p(AyBn) + p(AnBy) = p(ByAn) + p(BnAy)$
  - Not satisfied by most order effect models, but predicted by quantum
  - Can be tested experimentally.

Test of QQ equality

For the Clinton Gore discussed before:

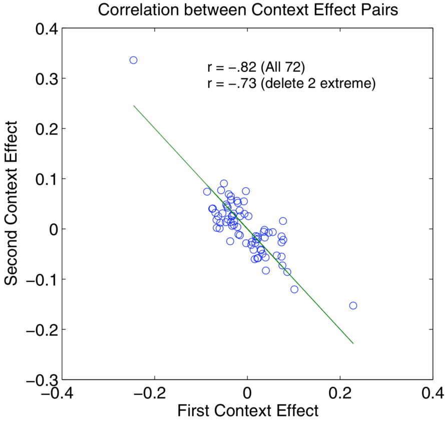
	Clinton–Gore	
	Gy	Gn
Cy	0.4899	0.0447
Cn	0.1767	0.2886
	Gore–Clinton	
	Gy	Gn
Cy	0.5625	0.0255
Cn	0.1991	0.2130
	Context effects	
	Gy	Gn
Cy	−0.0726	0.0192
Cn	−0.0224	0.0756

$\chi^2\,(3) = 10.14,\ p < 0.05$

$q = -0.003,\ \chi^2\,(1) = 0.01,\ p = 0.91$

Comprehensive study of QQ equality

70 national representative surveys, most containing more than 1,000 participants per survey (similar to Clinton-Gore), and 2 laboratory studies that manipulated question order.



Analysis of data

- Distributions of χ^2 statistics were analyzed for order effects and q values
- χ^2 distribution test for order effects produced a significant deviation from the null hypothesis (p = 0.0004)
- χ^2distribution test for the q values indicates no significant deviation from the null hypothesis (p = 0.4625)
- Across all 66 datasets, there are significant question order effects, and the QQ equality holds

Outline

What is quantum in SS? An example

- Should I buy a plot of land given the uncertainties due to the presidential elections?
- If Republican, I decide it is better to buy.
- If Democrat, I also decide it is better to buy.
- Therefore, I should prefer buying over not buying, even if I don't know who will win (Savage's Sure-thing Principle)

More formally.

- Consider the following to be true:
  - If A, then X is preferred over Y.
  - If $\neg A$, then X is preferred over Y.

More formally.

- Consider the following to be true:
  - If A, then X is preferred over Y.
  - If $\neg A$, then X is preferred over Y.
- Savage's Sure Thing Principle: X should be preferred over Y if we don't know whether A or $\neg A$.

Tversky and Shaffir.

Choice under risk. Won/Lost version.

- Imagine that you have just played a game of chance that gave you a 50% chance to win $200 and a 50% chance to lose $100. The coin was tossed and you have [won $200/lost $100]. You are now offered a second identical gamble
  - 50% chance to win $200 and
  - 50% chance to lose $100.
- Would you?
  - X: accept the second gamble? (69% if won (A), 59% if lost ($\neg A$)).
  - Y: reject the second gamble? (31% if won (A), 41% if lost ($\neg A$)).
- Clearly, X is preferred over Y regardless of condition A (won/lost).

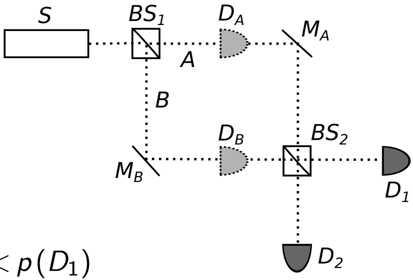
Tversky and Shaffir.

Choice under risk. Disjunctive version.

- Imagine that you have just played a game of chance that gave you a 50% chance to win $200 and a 50% chance to lose $100. Imagine that the coin has already been tossed but that you will not know whether you have won $200 or lost $100 until you make your decision concerning a second, identical gamble
  - 50% chance to win $200 and
  - 50% chance to lose $100.
- Would you?
  - X: accept the second gamble? (36%).
  - Y: reject the second gamble? (64%).

How to model with quantum?

Measurement of D_A or D_B affects D_1 and D_2.



Probabilities are non-monotonic:

$$p(D_1|D_A)\,p(D_A)+p(D_1|D_B)\,p(D_B)<p(D_1)$$

A possible model

- Consider the state

$$|\psi\rangle = c_G|\text{gamble}\rangle + c_{NG}|\text{no gamble}\rangle$$

as representing the

$$\psi_+ \approx 0.28 + e^{2.06i}0.12; \quad \psi_- \approx 0.65 + e^{1.89i}0.7.$$

- This fits the data.
- Problem: it only gives correct outcomes for "gamble" no "gamble" because it adds new parameters.

Outline

Pay-off matrix for PD

You defect	You cooperate
other defects	
other: 10	other: 25
you:10	you: 5
other cooperate	
other: 5	other: 20
you: 25	you: 20

Empirical observation

Shafir-Tversky (1992): 97 (know to defect); 84 (known cooperate); 63 (unknown)

Croson (1999): 67 (defect); 32 (cooperate); 30 (unknown)

Li & Taplan (2002): 83(defect); 66(cooperate); 60 (unknown);

Busemeyer et al. (2006): 91(defect); 84(cooperate); 66 (unknown);

Average: 84(defect); 66(cooperate); 55 (unknown)

Beliefs and actions

- $\Omega = \{B_D A_D, B_D A_C, B_C A_D, B_C A_C\}$, where A refers to your action and B to your belief of what the other person will do, and D and C are defect and cooperate.
- $$|\psi\rangle = \begin{bmatrix} \psi_{DD} \\ \psi_{DC} \\ \psi_{CD} \\ \psi_{CC} \end{bmatrix}.$$

Beliefs and actions

$$|\psi\rangle = \begin{bmatrix} \psi_{DD} \\ \psi_{DC} \\ \psi_{CD} \\ \psi_{CC} \end{bmatrix}.$$

- The terms for the Hamiltonian are (μ_D is a parameter for defection and μ_C for collaboration)

$$H_D = \frac{1}{\sqrt{1+\mu_D^2}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mu_D & 1 \\ 1 & -\mu_D \end{bmatrix}$$

$$H_C = \frac{1}{\sqrt{1+\mu_C^2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_C & 1 \\ 1 & -\mu_C \end{bmatrix}$$

Meaning of parameters

- $H_D + H_C$ rotates the state to favor either defection or cooperation, depending on the parameters μ_D or μ_C, corresponding to gain if defect
- $\mu_D = u(x_{DD} x_{DC})$ and $\mu_C = u(x_{CD} x_{CC})$, where x_{ij} is the payoff you receive if your opponent takes action i and you take action j
- u is a monotonic utility function

First term for the Hamiltonian

- The Hamiltonian is given by

$$H_0 = H_C + H_D = \begin{bmatrix} \frac{\mu_D}{\sqrt{1+\mu_D^2}} & \frac{1}{\sqrt{1+\mu_D^2}} & 0 & 0 \\ \frac{1}{\sqrt{1+\mu_D^2}} & -\frac{\mu_D}{\sqrt{1+\mu_D^2}} & 0 & 0 \\ 0 & 0 & \frac{\mu_C}{\sqrt{1+\mu_C^2}} & \frac{1}{\sqrt{1+\mu_C^2}} \\ 0 & 0 & \frac{1}{\sqrt{1+\mu_C^2}} & -\frac{\mu_C}{\sqrt{1+\mu_C^2}} \end{bmatrix}.$$

Dissonance

- H_0 is monotonic, and therefore does not violate STP
  - It is the "rational" part of the model
- Sometimes participants get new information that disagrees with their belief (when participants had to decide). So, H gets a correction

$$H_{NI} = \begin{bmatrix} -\frac{\gamma}{\sqrt{2}} & 0 & -\frac{\gamma}{\sqrt{2}} & 0 \\ 0 & \frac{\gamma}{\sqrt{2}} & 0 & -\frac{\gamma}{\sqrt{2}} \\ -\frac{\gamma}{\sqrt{2}} & 0 & \frac{\gamma}{\sqrt{2}} & 0 \\ 0 & -\frac{\gamma}{\sqrt{2}} & 0 & -\frac{\gamma}{\sqrt{2}} \end{bmatrix}$$

Final Hamiltonian

The final Hamiltonian is then

$$H_T = H_0 + H_{NI}$$

$$= \begin{bmatrix} \frac{\mu_D}{\sqrt{1+\mu_D^2}} - \frac{\gamma}{\sqrt{2}} & \frac{1}{\sqrt{1+\mu_D^2}} & -\frac{\gamma}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{1+\mu_D^2}} & -\frac{\mu_D}{\sqrt{1+\mu_D^2}} + \frac{\gamma}{\sqrt{2}} & 0 & -\frac{\gamma}{\sqrt{2}} \\ -\frac{\gamma}{\sqrt{2}} & 0 & \frac{\mu_C}{\sqrt{1+\mu_C^2}} + \frac{\gamma}{\sqrt{2}} & \frac{1}{\sqrt{1+\mu_C^2}} \\ 0 & -\frac{\gamma}{\sqrt{2}} & \frac{1}{\sqrt{1+\mu_C^2}} & -\frac{\mu_C}{\sqrt{1+\mu_C^2}} - \frac{\gamma}{\sqrt{2}} \end{bmatrix}$$

Evolution

- Evolution is given by

$$U(t) = e^{-iHt},$$

and t is chosen to be $\pi/2$ (roughly the average time participants make a decision).

- Initial state is

$$|\psi_0\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Choosing parameters $\mu_D = \mu_C = 0.59$, $\gamma = 1.74$ produces $(.68, .58, .37)$ whereas Tversky and Shaffir observed $(.69, .59, .36)$, a good fit.
- Other parameter values match well the other experiments.

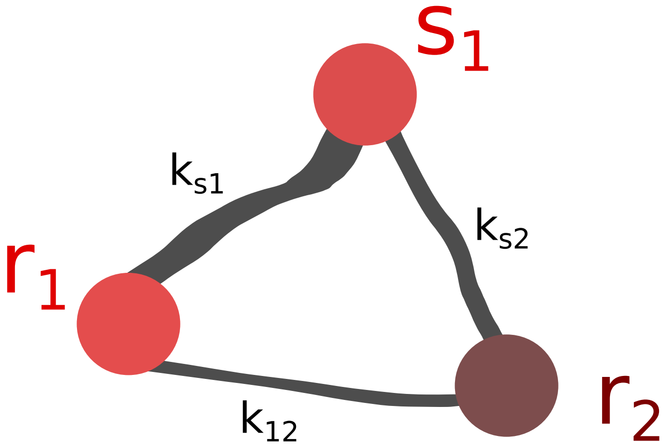
Outline

1. What is non-classical about quantum mechanics?
   - Non-determinism
   - Nonlocality
2. Contextuality, quantum, and probabilities
3. Quantum Cognition
   - Order Effect
   - The conjunction paradox and the two-slit
   - The prisoner's dilemma
   - A neurophysiological model for quantum cognition.
   - Beyond quantum cognition.

Neurons all the way down?

- What scale should we use?
  - Down to the synapse level?
  - Neurons?
  - Collective behavior of neurons?
- For language processing, robustness and measurable macroscopic effects suggest a *large* number of neurons.
- Even for a large collection of neurons, we still have several options with respect to modeling.
  - Do we need detailed interactions between neurons? Are the shapes of the action potential relevant? Timing?
- Our goal is to reduce the number of features, yet retain a physical meaning.

Stimulus and response neurons



Kuramoto Equations

- If no interaction,

$$O_i(t) = A_i \cos \varphi_i(t) = A_s \cos(\omega t),$$

$$\varphi_i = \omega_i t + \delta_i,$$

and

$$\dot{\varphi}_i = \omega_i.$$

Kuramoto Equations

- If no interaction,

$$O_i(t) = A_i \cos \varphi_i(t) = A_s \cos(\omega t),$$

$$\varphi_i = \omega_i t + \delta_i,$$

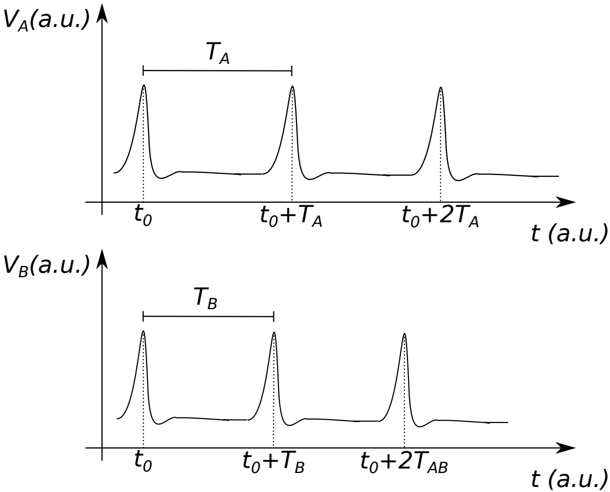
and

$$\dot{\varphi}_i = \omega_i.$$

- If we have a weak interaction, then

$$\dot{\varphi}_i = \omega_i - \sum_{j \neq i} A_{ij} \sin(\varphi_i - \varphi_j).$$

The intuition



How to represent responses with few oscillators?

- Each neural oscillator's dynamics can be described by the phase, φ.

$$s(t) = A_s \cos \varphi_s(t) = A_s \cos(\omega t),$$

$$r_1(t) = A_1 \cos \varphi_{r_1}(t) = A \cos(\omega t + \delta\varphi),$$

$$r_2(t) = A_2 \cos \varphi_{r_2}(t) = A \cos(\omega t + \delta\varphi - \pi).$$

$$I_1 \equiv \left\langle (s(t) + r_1(t))^2 \right\rangle_t = A^2 (1 + \cos(\delta\varphi)).$$

$$I_2 \equiv \left\langle (s(t) + r_2(t))^2 \right\rangle_t = A^2 (1 - \cos(\delta\varphi)).$$

- A response is the balance between the strengths I_1 and I_2,

$$b = \frac{I_1 - I_2}{I_1 + I_2} = \cos(\delta\varphi)$$

Encoding responses

- To encode responses, we need to modify

$$\dot{\varphi}_i = \omega_i - \sum_{j \neq i} A_{ij} \sin\left(\varphi_i - \varphi_j\right)$$

to include angles, i.e.,

$$\dot{\phi}_i = \omega_i + \sum A_{ij} \sin\left(\phi_j - \phi_i + \delta\varphi_{ij}\right).$$

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Encoding responses

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to include angles, i.e.,

$$\dot{\phi}_i = \omega_i + \sum A_{ij} \sin(\phi_j - \phi_i + \delta\varphi_{ij}) .$$

$$\dot{\phi}_i = \omega_i + \sum [A_{ij} \sin(\phi_j - \phi_i) + B_{ij} \cos(\phi_j - \phi_i)] .$$

Reinforcing oscillators

- During reinforcement:

$$\begin{aligned}\dot{\phi}_i &= \omega_i + \sum \left[A_{ij} \sin\left(\phi_j - \phi_i\right) + B_{ij} \cos\left(\phi_j - \phi_i\right)\right] \\ &\quad + K_0 \sin\left(\varphi_E - \varphi_i + \delta_{Ei}\right).\end{aligned}$$

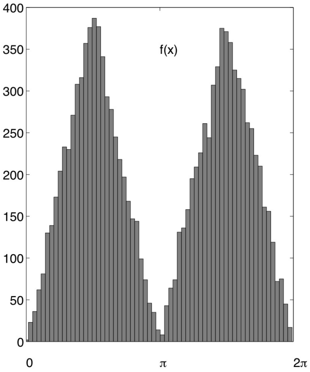
$$\frac{dk_{ij}^E}{dt} = \varepsilon\left(K_0\right)\left[\alpha \cos\left(\varphi_i - \varphi_j\right) - k_{ij}\right],$$

$$\frac{dk_{ij}^I}{dt} = \varepsilon\left(K_0\right)\left[\alpha \sin\left(\varphi_i - \varphi_j\right) - k_{ij}^I\right].$$

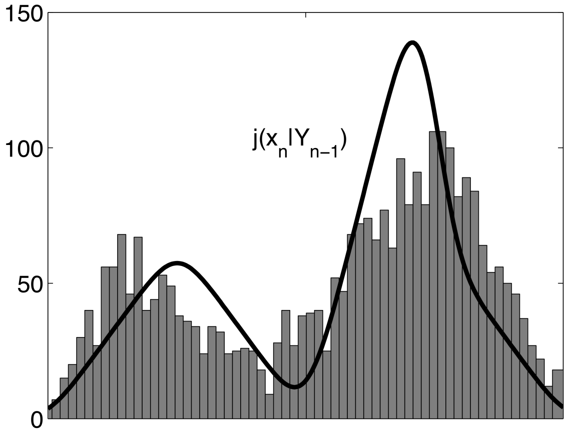
Recapping

- We represent a collection of neurons by the phase of their coherent oscillations.
- The phase difference between stimulus and response oscillators encode a continuum of responses.
- The dynamics comes from inhibitory as well as excitatory neuronal connections.

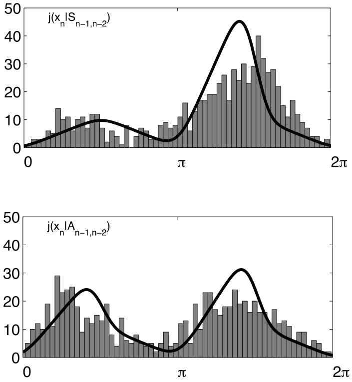
Response selection



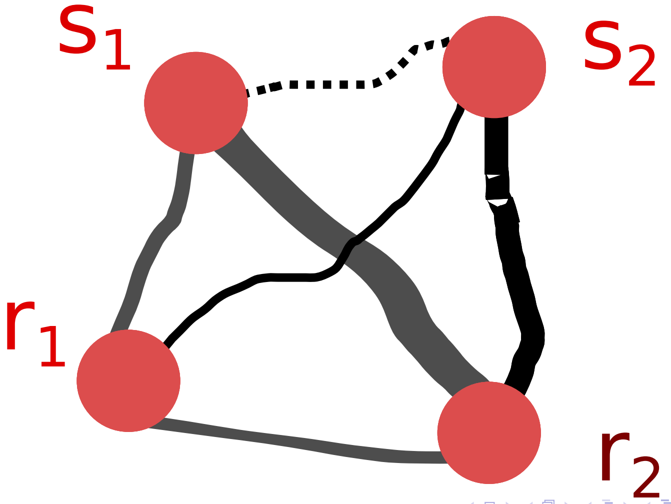
Conditional probabilities



Conditional probabilities



Oscillator interference



Some data

- For two stimulus oscillators, s_1 and s_2, and two response oscillators, r_1 and r_2.
- We select couplings between oscillators such that X is selected 60% of the time if s_1 is active, and 50% of the time if s_2 is active.
- By selecting the couplings between s_1 and s_2, we can control the degree of synchronicity between then.
- If s_1 and s_2 are activated, we can have interference between s_1 and s_2.
- In such cases, X is selected less than 40% of the time.

What the \#\$ *! do we know!?

- Propagation of oscillations on the cortex behave like a wave.
- Neural oscillator interference may be sensitive to context.
- Could quantum effects be simply contextual?

Outline

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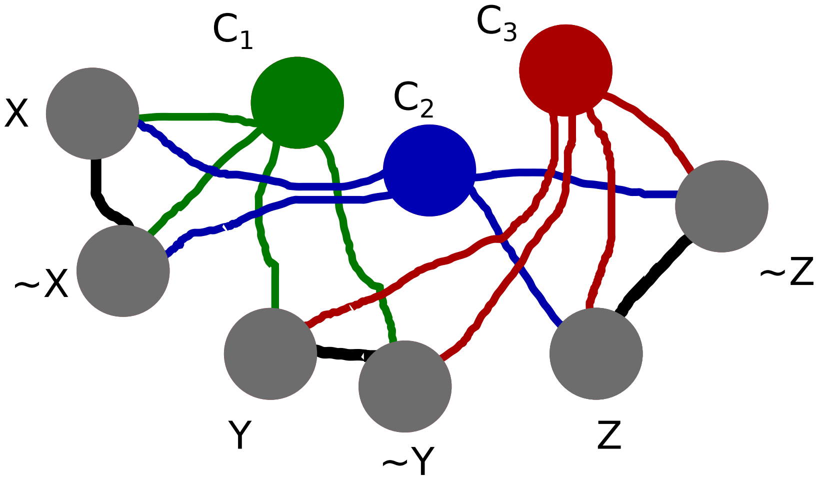
The simplest example

- Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be ± 1 random variables with zero expectation.
- Let
$$E(\mathbf{XY}) = E(\mathbf{YZ}) = E(\mathbf{XZ}) = \varepsilon.$$
- $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ have a joint probability distribution if and only if $\varepsilon > -1/3$.
- This is the simplest example of a set of random variables without a joint probability.

Can it make sense?

- It is possible to give a (albeit contrived)example where **X**, **Y**, and **Z** could not have a joint.
- Let each of the correlations, $E(\mathbf{XY}), E(\mathbf{YZ})$, and $E(\mathbf{XZ})$ correspond to different expert opinions which are inconsistent.
- Since the opinions are inconsistent, we don't have a joint.

A not-so-simple oscillator model



But it is not quantum!

- In their quantum version, we would have observables in a Hilbert space corresponding to each random variable **X**, **Y**, and **Z**. Call them $\hat{X}$, $\hat{Y}$, and $\hat{Z}$.
- To say that the correlations $E(\mathbf{XY}), E(\mathbf{YZ})$, and $E(\mathbf{XZ})$ have a certain value means that we can observe any pair of **X**, **Y**, and **Z**, i.e. $[\hat{X}, \hat{Y}] = [\hat{X}, \hat{Z}] = [\hat{Z}, \hat{Y}] = 0$.
- But the fact that they commute means we can find a basis where all operators $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are diagonal.
- Therefore, it is possible to measure simultaneously $\hat{X}$, $\hat{Y}$, and $\hat{Z}$, which means that there exists a joint probability distribution for **X**, **Y**, and **Z**.

Overall Summary.

- People behave in ways that are not rational, violating Kolmogorov's axioms of probability.
- The quantum mechanical formalism has been proposed as a tool in cognitive modeling (quantum cognition).
- Such formalism brings with it lots of baggage not present in brain processes (most notably non-locality as well as no signaling).
- If we relax the formalism, and allow simpler quantum-like interference (without the Hilbert space) with oscillators, we obtain systems that are not quantum.