Negative probabilities and counterfactual reasoning

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Outline

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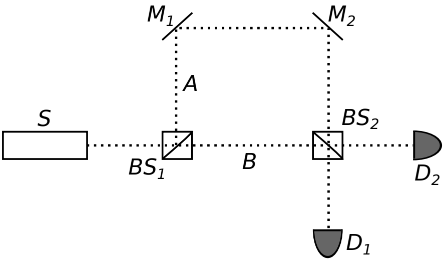
The mystery of QM (Feynman)

- The two-slit experiment has the essence of QM: it makes no sense.
- A simple version of the two-slit experiment is the Mach-Zehnder interferometer.



What does it mean to make no sense?

- It is non-monotonic.
- There is no joint probability distribution.
- We can't tell a story: consistent histories require proper measure.

What we cannot talk about we must pass over in silence

- We don't have consistent histories: so what can we talk about the two slit?
- Are there non-monotonic quasi-probabilities that we could use?
- If so, how do we interpret them?

Outline

Why Kolmogorov probabilities?

- Kolmogorov axiomatized probability in a set-theoretic way, with the following simple axioms.

$$\text{A1. } 1 \geq P(A) \geq 0$$

$$\text{A2. } P(\Omega) = 1$$

$$\text{A3. } P(A \cup B) = P(A) + P(B)$$

- Most ways to think rationally lead to probability measures a la Kolmogorov. For example,
  - Cox, Jaynes, Ramsey, de Finneti.
  - Venn, von Mises.

We can't always assign probabilities

There are systems that do not allow for a Kolmogorovian measure.

Example 1:

- $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are ± 1-valued random variables.

$$E(\mathbf{XY}) = E(\mathbf{XZ}) = E(\mathbf{YZ}) = -1.$$

- We can easily see why:

$$(\mathbf{X} = 1) \to (\mathbf{Y} = -1) \to (\mathbf{Z} = 1) \to (\mathbf{X} = -1).$$

Probabilities are monotonic

- Monotonicity is one a consequence of Kolmogorov's axioms.
- Say $C \subseteq D$. Then define $C' = D \setminus C$. Then C and C' are disjoint, and $C \cup C' = D$.
- But for disjoint sets

$$P\left(C \cup C'\right) = P(C) + P(C') = P(D).$$

From axiom A1 (positivity), it follows that if $C \subseteq D$ then $P(C) \leq P(D)$.

Double slit is not monotonic

- Feynman's remark on the two-slit are about the non-monotonicity of probabilities: how can we have a brighter spot when we actually close one of the slits?
- In other words, how can we have

$$P(A) > P(A \cup B)?$$

People reason nonmonotonically

- Busemeyer showed that we use nonmonotonic reasoning, as we violate monotonicity.
- Examples are
    - Conjunction fallacy
    - Violation of Savage's sure-thing principle (disjunction effect).

Upper and lower probabilities

- How do we get nonmonotonicity?
- de Finetti: relax Kolmogorov's axiom A2:

$$P^*(A \cup B) \geq P^*(A) + P^*(B)$$

or

$$P_*(A \cup B) \leq P_*(A) + P_*(B).$$

- Subjective meaning: bounds of best measures for inconsistent beliefs (imprecise probabilities).

Upper and lower probabilities

- Consequence:

$$M^* = \sum_i P_i^* > 1,$$

$$M_* = \sum_i P_{*i} < 1.$$

- M^* and M_* should be as close to one as possible.
- Inequalities and nonmonotonicity make it hard to compute upper and lowers for practical problems.

Workaround?

- Instead of violating A2, let us relax A1.

$$\text{A1.}\;\; M^T = \sum_i |p_i| \;\text{ is minimum}$$

$$\text{A2.}\;\; p(A \cup B) = p(A) + p(B)$$

$$\text{A3.}\;\; p(\Omega) = 1.$$

- p_i can now be negative.
- p defines an optimal upper probability distribution by simply setting all negative probability atoms to zero.
- Atoms with negative probability are thought subjectively as impossible events.

Why negative probabilities?

- We can compute them easily (compared to uppers/lowers).
- May be helpful to think about certain contextual problems (Oas and Al-Safi talks on Monday).
- They have a meaning in terms of subjective probability.

Outline

What doesn't work?

- Let us start with a simple way to think about the Mach-Zehnder interferometer.
- For the Mach-Zehnder, we have four possible experimental conditions:
  - D_1, D_2 only.
  - D_1, D_2, D_A.
  - D_1, D_2, D_B.
  - D_1, D_2, D_A, and D_B.

D_1, D_2 only. We have that $p_{d_1\overline{d}_2} = 1$ and $p_{d_1 d_2} = p_{\overline{d}_1 d_2} = p_{\overline{d}_1\overline{d}_2} = 0$.

D_1, D_2, D_A. Detection on D_A implies no detection on D_1 and D_2; no detection on D_A implies D_1 and D_2 are equiprobable.

$p_{d_a d_1 d_2} = p_{d_a d_1 \overline{d}_2} = p_{d_a \overline{d}_1 \overline{d}_2} = p_{d_a \overline{d}_1 d_2} = 0$, $p_{\overline{d}_a d_1 d_2} = p_{\overline{d}_a \overline{d}_1 \overline{d}_2} = 0$, and $p_{\overline{d}_a d_1 \overline{d}_2} = p_{\overline{d}_a \overline{d}_1 d_2} = \frac{1}{2}$.

D_1, D_2, D_B. $p_{d_b d_1 d_2} = p_{d_b d_1 \overline{d}_2} = p_{d_b \overline{d}_1 \overline{d}_2} = p_{d_b \overline{d}_1 d_2} = 0$, $p_{\overline{d}_b d_1 d_2} = p_{\overline{d}_b \overline{d}_1 \overline{d}_2} = 0$, and $p_{\overline{d}_b d_1 \overline{d}_2} = p_{\overline{d}_b \overline{d}_1 d_2} = \frac{1}{2}$.

D_1, D_2, D_A, and D_B.

$p_{d_a d_b d_1 d_2} = p_{d_a d_b \overline{d}_1 d_2} = p_{d_a d_b d_1 \overline{d}_2} = p_{d_a d_b \overline{d}_1 \overline{d}_2} = 0$, $p_{\overline{d}_a d_b d_1 d_2} = p_{\overline{d}_a d_b \overline{d}_1 d_2} = p_{\overline{d}_a d_b d_1 \overline{d}_2} = 0$, $p_{d_a \overline{d}_b d_1 d_2} = p_{d_a \overline{d}_b \overline{d}_1 d_2} = p_{d_a \overline{d}_b d_1 \overline{d}_2} = 0$, $p_{\overline{d}_a \overline{d}_b d_1 d_2} = p_{\overline{d}_a \overline{d}_b \overline{d}_1 d_2} = p_{\overline{d}_a \overline{d}_b d_1 \overline{d}_2} = p_{\overline{d}_a \overline{d}_b \overline{d}_1 \overline{d}_2} = 0$, and $p_{\overline{d}_a d_b \overline{d}_1 \overline{d}_2} = p_{d_a \overline{d}_b \overline{d}_1 \overline{d}_2} = \frac{1}{2}$.

What works?

- No probability can be defined (not even negative probabilities) for the case above.
- For the which path version, we observe

$$P(ab) = 0, \quad (1)$$

$$P(\bar{a}b) = \frac{1}{2}, \quad (2)$$

$$P(a\bar{b}) = \frac{1}{2}, \quad (3)$$

$$P(\bar{a}\bar{b}) = 0. \quad (4)$$

- Interference (maximum visibility) requires that we only observe in D_1:

$$P(d_1\overline{d}_2) = 1 \tag{5}$$

$$P(\overline{d}_1 d_2) = P(d_1 d_2) = P(\overline{d}_1\overline{d}_2) = 0. \tag{6}$$

- If we put a detector in $P_i \in \{A, B\}$, we "infer" that whenever we observe the particle *not* being in A, then the particle must be (probability 1) in B.
- But when we block the path, the probabilities are

$$P(\overline{a}d_1\overline{d}_2) = P(\overline{a}\overline{d}_1 d_2) = \frac{1}{2}, \tag{7}$$

$$P(\overline{b}d_1\overline{d}_2) = P(\overline{p}\overline{d}_1 d_2) = \frac{1}{2}. \tag{8}$$

- This is different from the previous example, where we talked about detections in A, and not lack of detections.

Negative joint: general solution

$$
\begin{aligned}
p_{abdd} &= -\frac{3}{4} + b, & p_{\bar{a}bdd} &= \frac{1}{2} - a + c - d,\\
p_{abd\bar{d}} &= \frac{3}{4} + c, & p_{\overline{ab}dd} &= b,\\
p_{a\overline{bd}d} &= a, & p_{a\bar{b}d\bar{d}} &= -c,\\
p_{\bar{a}b\overline{dd}} &= d, & p_{\overline{ab}d\bar{d}} &= c,\\
p_{a\bar{b}dd} &= \frac{1}{4} + a - 2b - c + d, & p_{ab\bar{d}d} &= \frac{1}{4} - a,\\
p_{\bar{a}b\bar{d}d} &= a, & p_{\bar{a}bd\bar{d}} &= -c,\\
p_{ab\overline{dd}} &= -\frac{1}{4} + a - b - c, & p_{a\overline{bdd}} &= \frac{1}{4} - 2a + 2b + 2c - d,\\
p_{\overline{abd}d} &= -a, & p_{\overline{ppdd}} &= a - b - c.
\end{aligned}
$$

Negative joint: a M^T minimizing solution

$$
\begin{array}{ll}
p_{abdd} = -\frac{3}{4}, & p_{\bar{a}bdd} = \frac{1}{2}, \\
p_{abd\bar{d}} = \frac{3}{4}, & p_{\overline{ab}dd} = 0, \\
p_{a\overline{bd}d} = 0, & p_{a\bar{b}d\bar{d}} = 0, \\
p_{\bar{a}b\overline{dd}} = 0, & p_{\overline{ab}d\bar{d}} = 0, \\
p_{a\bar{b}dd} = \frac{1}{4}, & p_{ab\bar{d}d} = \frac{1}{4}, \\
p_{\bar{a}b\bar{d}d} = 0, & p_{\bar{a}bd\bar{d}} = 0, \\
p_{ab\overline{dd}} = -\frac{1}{4}, & p_{a\overline{bdd}} = \frac{1}{4}, \\
p_{\overline{abd}d} = 0, & p_{\overline{ppdd}} = 0.
\end{array}
$$

Some conditionals

Since we have a "joint", we can compute conditional probabilities. E.g.

$$p(a|d_1\overline{d_2}) = \frac{1}{3},$$

$$p(b|d_1\overline{d_2}) = \frac{2}{3}.$$

To summarize:

- Random variables with inconsistent correlations can be thought in terms of negative probabilities. Such probabilities can be given a subjective interpretation in terms of upper bounds for imprecise subjective probabilities.
- In the two slit experiment, by reducing the amount of counterfactuals, it is possible to obtain a negative joint probability for the atoms.
- Such negative joint probability can then be used to compute unmeasured correlations of conditional probabilities, consistent with the marginals.

Thank you!