Unifying two methods of measuring quantum contextuality

J. Acacio de Barros[1], E. Dzhafarov[2], J. Kujala[3], and G. Oas[4]

[1]San Francisco State University, [2]Purdue University, [3]University of Jyväskylä, [4]Stanford University

June 9, 2014

Outline

1. Motivation
2. Leggett-Garg
3. Bell-EPR

Outline

1. Motivation

2. Leggett-Garg

3. Bell-EPR

Mystery of QM

- Feynman: the mystery of QM is in the two slit experiment.
  - Non-monotonic ($P(A \cup B) < P(A)$).
  - Bohm (explicitly contextual).
- Contextuality encompasses the mystery of quantum mechanics:
  - Leggett-Garg: contextuality in time of macroscopic systems.
  - Bell-EPR: superluminal contextuality.
  - Kochen-Specker: no noncontextual hidden variable.

What is contextuality

- $(\mathbf{P}, \mathbf{Q}, \mathbf{R}, \ldots)$ are jointly recorded under certain conditions C, and $(\mathbf{P}, \mathbf{Q}', \mathbf{R}', \ldots)$ are jointly recorded under different conditions C'.
- Contextuality is nonexistence of a joint distribution $(\mathbf{P}, \mathbf{Q}, \mathbf{R}, \ldots, \mathbf{Q}', \mathbf{R}', \ldots)$.
- Conditions C and C' may create different (and irreconcilable) contexts for the variable $\mathbf{P}$.

Looking at contextuality from two different views

- We explore contextuality from two different angles:
    - negative probabilities (non-monotonic)
    - explicit use of contextual variables
- Two examples are going to be presented:
    - Leggett-Garg
    - Bell-EPR

Outline

1 Motivation

2 Leggett-Garg

3 Bell-EPR

Leggett-Garg variables

- For a given *macroscopic* quantum system, there is a measurable quantity, $Q(t)$.
- Let $\mathbf{Q}(t)$ be a dichotomic random variable corresponding to the observed value of Q at time t.
- There are times $t_1 < t_2 < t_3$, such that

$$\mathbf{Q}(t_1), \mathbf{Q}(t_2), \mathbf{Q}(t_3),$$

  do not have a joint probability distribution.
- Contexts are: $\mathbf{Q}_1\mathbf{Q}_2$, $\mathbf{Q}_2\mathbf{Q}_3$, and $\mathbf{Q}_1\mathbf{Q}_3$.

Negative Probability approach

- Relax requirement that $1 \geq P(A) \geq 0$, allowing for a P consistent with marginals.
- Minimize the L1 norm of P:

$$M = \sum_{\omega_i \in \Omega} |P(\{\omega_i\})|.$$

- M^*, the minimum value, is the probability mass.

Meaning of M^*

- Minimizing the $L1$ distance makes the negative probability as close as possible to a proper probability p:
  - if $M^* = 1$, P is a standard probability measure.
  - if $M^* > 1$, we have no proper joint probability distribution (neg prob).
  - The greater the value of M^*, the further apart it is from a proper joint.
- Thus, M^* can be thought as a measure of contextuality.

M^* for Leggett-Garg

- There are 2^3 possibilities for $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$: $(1, 1, 1)$, $(1, 1, -1)$, ..., $(-1, -1, -1)$
- For correlations, M is given by

$$M \geq \frac{1}{2} + \frac{1}{2} s_1 \left(\langle \mathbf{Q}_1 \mathbf{Q}_2 \rangle , \langle \mathbf{Q}_1 \mathbf{Q}_3 \rangle , \langle \mathbf{Q}_2 \mathbf{Q}_3 \rangle \right),$$

$$M \geq 1,$$

with

$$s_1(a_1, \ldots, a_n) = \max_{\text{odd number of -'s}} \{ \pm a_1 \cdots \pm a_n \} .$$

- Recall $M = 1$ is necessary condition for existence of a joint.

Symmetric case

- From the above inequality,

$$M^* = \max\left\{1, \frac{1+s_1(\langle \mathbf{Q}_1\mathbf{Q}_2\rangle, \langle \mathbf{Q}_1\mathbf{Q}_3\rangle, \langle \mathbf{Q}_2\mathbf{Q}_3\rangle)}{2}\right\},$$

- For simplicity of analysis, we take the symmetric case

$$\langle \mathbf{Q}_1\mathbf{Q}_2\rangle = \langle \mathbf{Q}_2\mathbf{Q}_3\rangle = \langle \mathbf{Q}_1\mathbf{Q}_3\rangle = \varepsilon.$$

- Clearly, no joint when $\varepsilon = -1$, and

$$M^* = \begin{cases} 1, & -1/3 \le \varepsilon \le 1, \\ 1/2\left(1-3\varepsilon\right), & -1 \le \varepsilon \le -1/3. \end{cases}$$

Contextuality by Default approach

- Six random variables

$$\mathbf{Q}_{12}, \mathbf{Q}_{13}, \mathbf{Q}_{21}, \mathbf{Q}_{23}, \mathbf{Q}_{31}, \mathbf{Q}_{32}.$$

- $\mathbf{Q}_{12}$ represents $\mathbf{Q}_1$ under the recording condition $\mathbf{Q}_2$, etc.

$$(\mathbf{Q}_{12}, \mathbf{Q}_{21}), (\mathbf{Q}_{13}, \mathbf{Q}_{31}), (\mathbf{Q}_{23}, \mathbf{Q}_{32}).$$

- Clearly, there is always a joint.

Symmetric case

- If it is possible to have

$$P(\mathbf{Q}_{12} = \mathbf{Q}_{13}) = P(\mathbf{Q}_{21} = \mathbf{Q}_{23}) = P(\mathbf{Q}_{31} = \mathbf{Q}_{32}) = 1,$$

  then variables are not contextual.
- Let

$$\langle \mathbf{Q}_{12}\mathbf{Q}_{13} \rangle = \langle \mathbf{Q}_{21}\mathbf{Q}_{23} \rangle = \langle \mathbf{Q}_{31}\mathbf{Q}_{32} \rangle = 1 - \alpha$$

- The minimum value of α gives a measure of contextuality.

$$\alpha_{\text{min}} = \begin{cases} 0, & -1/3 \leq \varepsilon \leq 1, \\ -1/3 - \varepsilon, & -1 \leq \varepsilon \leq -1/3. \end{cases}$$

Summary of Leggett-Garg

$$M^* = \begin{cases} 1, & -1/3 \leq \varepsilon \leq 1, \\ 1/2\,(1 - 3\varepsilon)\,, & -1 \leq \varepsilon \leq -1/3. \end{cases}$$

$$\alpha_{\text{min}} = \begin{cases} 0, & -1/3 \leq \varepsilon \leq 1, \\ -1/3 - \varepsilon, & -1 \leq \varepsilon \leq -1/3. \end{cases}$$

$$\alpha_{\text{min}} = \frac{2}{3}\left(M^* - 1\right).$$

Outline

1 Motivation

2 Leggett-Garg

3 Bell-EPR

M^* and $\alpha_{\min}$

$$M^* = \max\left\{1, \frac{s_1(\langle \mathbf{A}_1\mathbf{B}_1\rangle, \langle \mathbf{A}_1\mathbf{B}_2\rangle, \langle \mathbf{A}_2\mathbf{B}_1\rangle, \langle \mathbf{A}_2\mathbf{B}_2\rangle)}{2}\right\}$$
$$= \max\left\{1, \frac{S}{2}\right\},$$

$$\alpha_{\min} = \max\left\{0, \frac{s_1\left(\langle \mathbf{A}_{11}\mathbf{B}_{11}\rangle, \langle \mathbf{A}_{12}\mathbf{B}_{12}\rangle, \langle \mathbf{A}_{21}\mathbf{B}_{21}\rangle, \langle \mathbf{A}_{22}\mathbf{B}_{22}\rangle\right) - 2}{4}\right\}.$$

Symmetric case

- Let
$$\langle \mathbf{A}_1\mathbf{B}_1\rangle = \langle \mathbf{A}_1\mathbf{B}_2\rangle = \langle \mathbf{A}_2\mathbf{B}_1\rangle = -\langle \mathbf{A}_2\mathbf{B}_2\rangle = \varepsilon.$$
- $S = 4\varepsilon$, and no joint exists if $S > 2$; S from CHSH inequalities.
- For this case,
$$M^* = \begin{cases} 1, & |\varepsilon| \leq 1/2, \\ 2|\varepsilon|, & |\varepsilon| > 1/2, \end{cases}$$
$$\alpha_{\text{min}} = \begin{cases} 0, & |\varepsilon| \leq 1/2, \\ |\varepsilon| - 1/2, & |\varepsilon| > 1/2. \end{cases}$$
or
$$\alpha_{\text{min}} = \frac{1}{2}(M^* - 1).$$

Final comments

- Two conceptually different measures of contextuality:
  - Minimum total negative probability mass, M^*.
  - Minimum deviation from another context, α_{min}.
- M^* and α_{min} are linearly related for both.
- For Bell-EPR, $M^* = S/2$, $\alpha_{\text{min}} = \frac{1}{4}(S-2))$.
- Provide interchangeable measures of contextuality for non-signaling systems.

Thank you!