

A Suppes Predicate for General Relativity and Set-Theoretically Generic Spacetimes

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We summarize ideas from Zermelo-Fraenkel set theory up to an axiomatic treatment for general relativity based on a Suppes predicate. We then examine the meaning of set-theoretic genericity for manifolds that underlie the Einstein equations. A physical interpretation is finally offered for those set-theoretically generic manifolds in gravitational theory.

1. INTRODUCTION

The discovery of exotic differentiable structures on \mathbf{R}^4 (Taubes, 1987; Kirby, 1989; Freed and Uhlenbeck, 1985) leads to the still open question of their physical interpretation. The relation between “exotic differentiable structures” and their “physical interpretation” is the same as the relation between a given syntactic structure and its underlying semantics. However, since the plethora of differentiable structures for \mathbf{R}^4 is a *theorem* in differential geometry, when we formalize that discipline (and, of course, when we identify general relativity constructs with geometric constructs) within a standard axiomatic system such as the Zermelo-Fraenkel system, the wealth we get at the syntactic level is exactly mirrored at the semantic level, that is, what is logically valid at the syntactic level is true in all models for that structure.

We discuss in the present paper a different kind of wealth of new objects that appear in the interplay between general relativity’s syntax and

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its underlying semantics, namely that given to us by the use of forcing models for segments of the Zermelo–Fraenkel system. We describe here a simple axiomatic treatment for general relativity along the lines suggested by Suppes (1967) and elaborate on some of its forcing models. Suppes (1967) suggested that in order to axiomatize a theory—any mathematically formulated theory—we must define a set-theoretic predicate. A set-theoretic predicate is simply a predicate in the formal language of a given axiomatic set theory. We thus have a nice and convenient recipe for the construction of axiomatized versions for many theories in the realm of natural science, granted that they can be given a precise mathematical formulation, as all of everyday mathematics can be formulated within an established axiomatic set theory such as the Zermelo–Fraenkel system.

Da Costa and Chuaqui (1988) related Suppes predicates to the Bourbaki (1957) structure concept, and da Costa and Doria (1989*b*) gave a summary axiomatic treatment for most of classical (or first-quantized) physics. Now our main motivation for the development of a convenient axiomatic treatment for physics with the help of Suppes predicates lies in the huge multiplicity of models for the Zermelo–Fraenkel axioms: a glimpse at what can happen when we move from one model into another when dealing with an axiomatized physical theory has been given by da Costa and Doria (1989*a*) and Barros (1989). We wish to start in this paper a more systematic exploration of the meaning of concepts like set-theoretic genericity and similar forcing-related notions in physics.

We restrict our attention to differentiable manifolds that may support (may underlie) the Einstein gravitational field equations. Section 2 is a resumé of the main concepts we need from axiomatic set theory up to Suppes predicates in the da Costa–Chuaqui version: we also present in that section our own version for the axiomatics of general relativity. Section 3 discusses cylindrical support manifolds for spacetimes, that is, 4-manifolds of the form $C \times \mathbf{R}$, where \mathbf{R} are the reals and C is a compact smooth 3-manifold. There, we show the following: all such manifolds are (set-theoretically) standard, that is, no new cylindrical spacetime supports are added to our theory when we move from a given model for the Zermelo–Fraenkel axioms into any larger forcing extension of that model.

The situation is different when we turn to the noncompact case: here entirely new set-theoretically generic spacetime supports appear when we make adequate forcing extensions. That case is discussed in Section 4, where we also argue that the new spacetimes which are introduced by forcing are *physically* different from those in the original model. Section 5 ponders the relation between general covariance and set-theoretic genericity for our manifolds; it is seen that (modulo the action of local diffeomorphisms) open balls are standard, so that set-theoretical genericity in the case of

noncompact support manifolds appears as an essentially *global* property. (That fact is clear even in the case of generic *exotic* manifolds homeomorphic to \mathbf{R}^4 .) However, the interplay between “genericity” and “globality” is not a simple one, as shown in the several examples which are introduced and discussed in that section. Section 6 elaborates on the space \mathcal{S} of all spacetime supports. We show in that section that we cannot *formally prove* within ZFC that set-theoretical genericity and randomness coincide, modulo a meager set, when dealing with the objects in \mathcal{S} . However, we can convincingly and rigorously argue in an informal way that both properties do coincide, but for a meager set. The physical considerations that arise out of our discussion are evaluated in Section 7.

The present paper stems from a suggestion of Cohen and Hersh (1967), where they notice that forcing models (and other metamathematical techniques) may have the same import to physics as non-Euclidean geometries. Papers in that direction are Benioff (1976*a,b*) as well as Augerstein (1984), Ross (1984), Chaitin (1982), da Costa and Doria (1989*a*), and Barros (1989).

2. A BRIEF SURVEY OF AXIOMATIC SET THEORY

We summarize here the main tools we require from axiomatic set theory and from the theory of structures. Complete references are given at the end of the section.

2.1. Zermelo–Fraenkel Set Theory

Our work is done within pretty conventional mathematics, that is, we use here the Zermelo–Fraenkel set theory together with the Axiom of Choice (ZFC). ZFC is built upon a first-order logic that formalizes classical predicate logic with equality. Its axioms are as follows.

ZFC1 *Extensionality.* Two sets are equal if and only if they have the same elements.

ZFC2 *Pair.* Given two sets x and y , there exists the set $\{x, y\}$ with x and y as its sole elements.

ZFC3 *Union.* Let x be a set of sets. There exists a set $\cup x$ whose elements are all elements of the sets in x .

ZFC4 *Power set.* There is a set whose elements are all the subsets of a given set. If x is such a set, the set of all its subsets is denoted $\mathcal{P}x$, and is called the power set of x .

ZFC5 *Separation.* Let y be a set, and let P be a property formulated in the language of ZFC. Then there is a set x whose elements are precisely the members of y that satisfy P .

ZFC6 *Infinity.* There is a set that contains all the natural numbers.

ZFC7 Replacement. Let $F(x, y)$ be a formula in the language of ZFC, where x and y are free variables and for each x there is at most one y so that $F(x, y)$. The collection of all sets y that satisfy $F(x, y)$ is a set. In other words, $\{y: F(x, y) \text{ is true and } x \text{ is a set}\}$ is again a set.

ZFC8 Choice. Let x be a nonempty set whose members are again nonempty sets. Then there is a set y containing one and only one element from every member of x .

ZFC9 Foundation. Every set can be obtained from the empty set \emptyset with the help of set-theoretic operations.

With separation and extensionality we prove the existence and unicity of the empty set \emptyset . Union, separation, and power set allow us to collect members of a previously given set in order to form a new set; the restriction thus imposed on union and separation avoids well-known paradoxes like Russell's. We can get all the usual mathematical notions within ZFC: relations, functions; ordinals and cardinals; the natural numbers; integers, rationals, and reals; the complex field; and so on. Algebraic and topological constructs are also easily formalized within ZFC, so that we get axiomatized versions for the whole of classical analysis, functional analysis, differential geometry and topology, and algebraic topology. The Axiom of Choice allows us to have some of the most powerful tools in classical mathematics: the Hahn–Banach theorem in functional analysis; the Gel'fand–Naimark–Segal theorem in the theory of Hilbert space rings of operators; the Tychonov theorem in topology; the theorem that asserts the existence of nonmeasurable sets on the real line with respect to Lebesgue measure—and its beautiful consequence, the so-called Banach–Tarski “paradox”; and so on.

2.2. Models

Some collections of ZFC sets are not sets; those collections are called *proper classes* (sets are *improper* classes). A class x —in particular a set—is transitive if, whenever $y \in x$, then $y \subset x$, that is, every element of y is also an element of x . A *model* for ZFC is a class where all ZFC axioms are satisfied. If the model is a set, it is a *small model*; otherwise it is a *large model*. A first-order theory such as ZFC based on classical logic is consistent if and only if it has a model. However, due to Gödel's incompleteness theorem, we cannot prove *within* ZFC that it has a small model, but we can show it for any finite set of sentences of ZFC (notice that the number of axioms for ZFC is denumerably infinite, as ZFC5 and ZFC7 are axiom schemata). *Standard models* for ZFC are transitive models where the membership relation interprets \in . We will only deal with standard models. If **ORD** denotes the proper class of ordinal numbers, and if lower-case Greek

letters refer to ordinals, then the proper class V given by the recursion below is the well-founded ZFC universe. V is the universal proper class:

$$V_0 = \emptyset \tag{2.1}$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha) \tag{2.2}$$

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta, \quad \text{where } \alpha \text{ is a limit ordinal} \tag{2.3}$$

$$V = \bigcup_{\alpha \in \text{ORD}} V_\alpha \tag{2.4}$$

Given any set x , there is a minimum ordinal α such that $x \in V_\alpha$; α is the *rank* of x , and we write $\alpha = \text{rank}(x)$. We can also form a hierarchy over a prescribed set x ,

$$V_0 = x, \quad x \text{ a set} \tag{2.5}$$

$$V_{\alpha+1} = \mathcal{P}V_\alpha(x) \tag{2.6}$$

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta(x), \quad \alpha \text{ a limit ordinal} \tag{2.7}$$

$$V(x) = \bigcup_{\alpha \in \text{ORD}} V_\alpha(x) \tag{2.8}$$

If α is the least ordinal so that $y \in V_\alpha(x)$, then α is the *rank of y relative to x* .

Now let x be a ZFC set and let $y \subset x$. y is *definable* from x if and only if there is a formula $F(t_1, \dots, t_n, z)$, the $t_1, \dots, t_n \in x$ and kept fixed, so that z ranges precisely over the elements of y . We write $w = \Pi(x)$ to abbreviate “ w is the set of subsets of x which are definable from x .” By induction as above we have

$$L_0 = \emptyset \tag{2.9}$$

$$L_{\alpha+1} = \Pi(L_\alpha) \tag{2.10}$$

$$L_\alpha = \bigcup_{\beta < \alpha} L_\beta, \quad \alpha \text{ a limit ordinal} \tag{2.11}$$

$$L = \bigcup_{\alpha \in \text{ORD}} L_\alpha \tag{2.12}$$

L is the *Gödel constructible universe*; it is a proper class. We can also have a proper class $L(x)$ of all sets which are Gödel-constructible from x ; the properties of $L(x)$ and the axioms it satisfies will strongly depend on x .

Finally, if we add to ZFC a further axiom, the Standard Model Axiom, we may prove the existence of a small model for ZFC, called Cohen’s *minimal model*. The minimal model is countable, transitive, and Gödel-constructible.

2.3. Boolean-Valued Models

We are mainly interested in generic extensions of models for ZFC. We consider here forcing models and Boolean-valued models. Boolean-valued models are easier to understand; they are a variant of Cohen's forcing technique, and were developed in 1965–1967 by R. M. Solovay and D. S. Scott. Given a ZFC universe \mathbf{V} , let $B \in \mathbf{V}$ be a complete Boolean algebra (a Boolean algebra is complete when we can take arbitrary suprema and infima in it). The Boolean-valued universe is defined as follows:

$$\mathbf{V}^B = \emptyset \quad (2.13)$$

$$\mathbf{V}_{\alpha+1}^B = \mathbf{V}_\alpha^B \cup F_\alpha^B \quad (2.14)$$

$$\mathbf{V}_\alpha^B = \bigcup_{\beta < \alpha} \mathbf{V}_\beta^B, \quad \alpha \text{ a limit ordinal} \quad (2.15)$$

$$\mathbf{V}^B = \bigcup_{\alpha \in \mathbf{ORD}} \mathbf{V}_\alpha^B \quad (2.16)$$

F_α^B is the set of all *extensional functions* whose domains are contained in \mathbf{V}_α^B with values in B —a function y is extensional if, for $x \in \text{domain}(y)$, $y(x) = \|x \in y\|$, where $\|\cdot\cdot\cdot\|$ denotes the truth value of the expression $(\cdot\cdot\cdot)$. Boolean-valued models thus substitute a 2-valued truth-value system for a $|B|$ -valued system, where $|B|$ is the cardinality of the Boolean algebra. Moreover, one has $\mathbf{V}^B \models P$ if and only if $\|P\| = 1$, that is, P is true in \mathbf{V}^B if and only if its Boolean value is 1. Similarly, $\mathbf{V}^B \models \neg P$ if and only if $\|P\| = 0$, that is, P is false in \mathbf{V}^B if and only if its truth value is zero in \mathbf{V}^B .

2.4. Forcing

To see the relation between Boolean-valued models and Cohen's forcing models, we again start from a universe \mathbf{V} for the ZFC axioms and cut down from \mathbf{V} to \mathbf{M} , where \mathbf{M} is a countable transitive model for ZFC, with the help of the Löwenheim–Skolem theorem. Given an infinite set $\alpha \in \mathbf{M}$ (which will in general be an ordinal number), we form $\text{Fin}(\alpha, 2)$, the set of all maps from α to $2 = \{0, 1\}$ with finite domains. [$\text{Fin}(\alpha, 2) \in \mathbf{M}$, since finite objects are preserved.] An arbitrary map $g \in 2^\alpha$ may be pieced together from elements in $\text{Fin}(\alpha, 2)$, where we pick up finite sequences of 0's and 1's that fit inside g . That state of affairs has the following interpretation: each $p \in \text{Fin}(\alpha, 2)$ can be seen as coding some finite piece of information about a sentence in the language of ZFC. In our case, we can prove that $g \notin \mathbf{M}$. We can also show that for a convenient choice of g the truth or falsity of all sentences in our formal language is eventually decided by a

given p in g and by all its extensions in g . We then write “ $p \Vdash P$,” that is, the sequence p “forces” the assertion P , and $g = \cup p$, where g contains all information that can be consistently pieced up from the p ’s.

Now, $\text{Fin}(\alpha, 2)$ can be densely embedded in a 1-1 way into a complete Boolean algebra B . We topologize $\text{Fin}(\alpha, 2)$ with its natural order topology (which is here induced from B), and consider a countable set of its dense subsets that are also in the countable model \mathbf{M} . We require that the map g , partitioned into all its finite pieces p , meet each one of those dense subsets. Then g will be completed in B up to a *generic* ultrafilter $U(g) \subset \mathcal{P}(B)$. Given the embedding $\text{Fin}(\alpha, 2) \subset B$, we then form the Boolean extension \mathbf{M}^B and define in that extension “ $p \Vdash P$ if and only if $p \leq \|P\|$.”

To get the forcing extension, we define the equivalence relation $x \sim y$ if and only if $\|x = y\| \in U(g)$, and $x \in_U y$ if and only if $\|x \in y\| \in U(g)$. The quotient $\mathbf{M}^B / U(g)$ via those relations is a model for the ZFC axioms, as \mathbf{M}^B and \mathbf{V}^B . From that quotient we then get by Mostowski collapse a countable transitive model $\mathbf{M}(g)$ isomorphic to $\mathbf{M}^B / U(g)$, which is the forcing extension of \mathbf{M} with respect to g . Everything that is true in \mathbf{V}^B and in \mathbf{M}^B is true in $\mathbf{M}(g)$, but as one has to go through a quotient to get $\mathbf{M}(g)$, there are a few essential differences between both models. First, \mathbf{M}^B has in general an infinite set of truth values, while $\mathbf{M}(g)$ is a 2-valued model. As a result, we have Cohen’s truth lemma, “ $p \Vdash P$ if and only if $\mathbf{M}(g) \Vdash P$.” That result is not in general true of \mathbf{M}^B and \mathbf{V}^B ; also, $\mathbf{M}(g)$ is countable, while that condition is not required of \mathbf{V} and \mathbf{V}^B .

Forcing is more general than the technique of Boolean-valued models, since it can be applied to a smaller fragment of the ZFC axioms (in order to construct a Boolean algebra of high cardinality in \mathbf{V} , one requires the Power Set Axiom, while forcing can be applied with much simpler tools).

Finally, we notice that if $2 = \{0, 1\} \in B$ denotes the 2-element (complete) Boolean algebra, $\mathbf{V} = \mathbf{V}^2$, and that equality induces a natural embedding $\mathbf{V} \hookrightarrow \mathbf{V}^B$, where $x \in \mathbf{V} \mapsto \hat{x} \in \mathbf{V}^B$; \hat{x} is the *standard image* of x in \mathbf{V}^B .

2.5. Suppes Predicates

We axiomatize general relativity within ZFC with the help of Suppes predicates. A mathematical structure E is a finite ordered collection of sets of finite rank over the union of the ranges of two finite sequences of sets, x_1, x_2, \dots, x_m and y_1, y_2, \dots, y_n , where $m > 0$ and $n \geq 0$. Thus, E is a ZFC set. The x ’s are called *principal base sets*, while the y ’s are *auxiliary base sets*. A *Suppes predicate* is a formula of set theory whose only free variables are those shown:

$$P(E, x_1, \dots, x_m, y_1, \dots, y_n) \quad (2.17)$$

P is a conjunction of two parts: one specifies the set-theoretic construction of E out of the base sets, while the second part contains the axioms for the species of structures in which we are interested.

2.6. An Axiomatic Treatment for General Relativity

We sketch how one proceeds in the case of general relativity. We must at first formalize within ZFC the concept of spacetime, that is, a 4-dimensional real Hausdorff “smooth” manifold (“smooth” stands for C^k , $0 < k \leq +\infty$, Sobolev, or any differentiability condition on a function space), plus a Lorentzian metric—a symmetric 2-form, nowhere degenerate, with $a+2$ signature.

We start from the concept of “real number.” Given the empty set \emptyset , pair (ZFC2), power set (ZFC4), and separation (ZFC5) allow us to define the natural number sequence: $0 = \emptyset$, $1 = 0 \cup \{\emptyset\}$, $2 = 1 \cup \{\emptyset\}$, and so on. Then infinity (ZFC6) gives us ω_0 , the set of all natural numbers, and from the Cartesian product $\omega_0 \times \omega_0$ (obtained via union, separation, and power set), we get (via separation) the rational numbers \mathbf{Q} . Dedekind cuts (again power set) finally give us the real numbers \mathbf{R} , while a new Cartesian product (and separation) gives us the algebra of complex numbers.

Now a topological space is a pair $\langle X, T \rangle$, where X is an arbitrary set—anything in \mathbf{V} —and $T \subseteq \mathcal{P}(X)$ is the set of open sets in X (T includes \emptyset and X and is closed under arbitrary unions and finite intersections). The Suppes predicate for $\langle X, T \rangle$ tells us how to construct in ZFC that pair out of a previously given set X , together with the restriction on T given by the open set axioms.

We will also require algebraic structures, like those of group and vectorspace. For that of a vectorspace, one proceeds as follows: out of a given set X (the principal set) and the reals \mathbf{R} (the auxiliary set), we pick a distinguished element ($0 \in X$, the null vector), and on $X \times X$ and $\mathbf{R} \times X$, which are sets of finite rank over $X \cup \mathbf{R}$, we impose the usual axioms for a real vectorspace. Again the Suppes predicate tells us how to obtain $X \times X$ and $\mathbf{R} \times X$ out of $X \cup \mathbf{R}$ and subjects those sets to the corresponding vectorspace axioms.

A differentiable manifold can be built from $X \cup \mathbf{R}$, where X is a separable complete metric space (that is, a Polish space). We recover \mathbf{R} from $X \cup \mathbf{R}$ via separation; we then get \mathbf{R}^n , $n \in \omega_0$ kept fixed (n will be the manifold’s dimension). From power set we get the finite product sets \mathbf{R}^X and $\{\mathbf{R}^n\}^X$, whence we get by separation $K(X, \mathbf{R}) \subset \mathbf{R}^X$ and $K(X, \mathbf{R}^n) \subset (\mathbf{R}^n)^X$, where K denotes a particular smoothness criterion (K may be, say, C^∞ or Sobolev). We will also need the restrictions $K(U, \mathbf{R})$ and $K(U, \mathbf{R}^n)$, $U \subset M$, and the sets $K(X, X) \subset X^X$ and $K(\mathbf{R}^n, \mathbf{R}^n) \subset (\mathbf{R}^n)^{\mathbf{R}^n}$. Then the Suppes predicate for the species of structures of a differentiable manifold

formalizes in the language of ZFC the preceding constructions, together with the axioms that relate local domains on M to \mathbf{R}^n and to coordinate maps from \mathbf{R}^n to \mathbf{R}^n .

Now in order to get a spacetime, we take a real differentiable manifold M and form its tangent bundle $T \cdot M$ (which is set-theoretically constructed in ZFC through the usual equivalence classes of tangent curves on M at a given point) as well as its dual cotangent bundle $T^* \cdot M$. We then fix $n = 4$ and from the tensor product $\otimes T^* \cdot M$ we pick up a symmetric nowhere degenerate 2-cotensor g with a +2 signature. g is our spacetime's metric tensor. The imposition of a metric tensor g on M is equivalent to the requirement that one be given the following embedding: if $P(M, G)$ denotes a principal bundle over M with G , a finite-dimensional Lie group, as its fiber, and if $GL(4, \mathbf{R})$ is the 4-dimensional real linear group and if $O(3, 1)$ is the full Lorentz group, a +2 metric tensor on M is equivalent to the specification of a particular embedding $P(M, O(3, 1)) \subset P(M, GL(4, \mathbf{R}))$, M a spacetime manifold, where the latter bundle has as associated bundles the tensor bundles over M . We then get the couple $\langle M, g \rangle$, which will be our (formal) spacetime within ZFC. The Einstein equations, tensor fields, and the like will be cross sections of the several tensor bundles over $\langle M, g \rangle$.

The following references cover the material in this section: historical details can be found in Kneebone (1963) and Scott (1967, 1985). A general reference on logic and axiomatic set theory is Manin (1977). On set theory one might also cite Krivine (1969) and Cohen (1966); forcing is given in detail in Cohen (1966), Shoenfield (1971), Odifreddi (1983), and Kunen (1983). The full theory of Boolean-valued universes and its relation to forcing is seen in Bell (1985); the main results we have used from differentiable geometry are in Kobayashi and Nomizu (1963). Finally, our presentation of the axiomatics of general relativity is based on da Costa and Chuaqui (1988), da Costa and French (1990), and da Costa and Doria (1989*b*).

Set-theoretic notations follow Kunen (1983) and Bell (1985); on geometry we follow Kobayashi and Nomizu (1963). When the same kind of notation is used for different objects in our exposition, the distinction will be made clear from context, or we will momentarily change the notation to suit our purposes.

3. CYLINDRICAL SPACETIMES IN BOOLEAN-VALUED UNIVERSES

Our main interest lies in the behavior of differentiable 4-manifolds M that are carriers of a spacetime structure $\langle M, g \rangle$ in the sense of general relativity within a Boolean-valued or forcing extension of the ZFC well-founded universe \mathbf{V} or of Gödel's constructible universe \mathbf{L} .

Almost all our theorems will either be ZFC theorems, and as such will be prefixed “ZFC $\vdash \dots$,” which means, “it may be deduced from the ZFC axioms that \dots ,” or will be valid assertions within a particular model for a given set of axioms; if \mathbf{M} is that model, we put “ $\mathbf{M} \models \dots$,” which means, “it is true in \mathbf{M} that \dots .” This means that all our objects are found within some particular ZFC universe.

3.1. Cylindrical Spacetime Supports Form a Countable Set

Let C be a compact Hausdorff real, smooth 3-manifold. A *cylindrical spacetime support* M is any manifold diffeomorphic to $C \times \mathbf{R}$ (when endowing M with a Lorentzian metric tensor one usually requires that the cross section $C \times \{x\}$, $x \in \mathbf{R}$, be spacelike, but that condition plays no part in our present discussion). We make precise our concept of “smoothness” in what follows. Also, in order to ensure that discs have an adequately smooth boundary, we suppose that \mathbf{R}^n is endowed with the Euclidean metric.

We need a counting result:

Proposition 3.1. ZFC \vdash The set of all diffeomorphism classes of cylindrical spacetime supports is an infinite countable set.

Proof. First we show that there are *at least* \aleph_0 compact 3-manifolds. There are exactly \aleph_0 compact 2-manifolds (for the classification theorem see Massey (1967) and Novikov *et al.* (1987)). Now if N is a compact 2-manifold, $N \times S^1$ is a compact 3-manifold, where S^1 is the 1-sphere. So, there are at least \aleph_0 compact 3-manifolds modulo diffeomorphisms. ■

Now we show that there are *at most* \aleph_0 compact 3-manifolds modulo diffeomorphisms. Actually we prove two results: first we notice the following.

Lemma 3.2. ZFC \vdash Every C^∞ differentiable finite-dimensional real manifold is triangulizable.

Proof. See Whitehead (1940). ■

Since we deal with compact manifolds, we proceed as follows: a given manifold’s triangulation is denoted

$$K = \langle \langle 0, 1, \dots, n \rangle, \langle \sigma_1, \sigma_2, \dots, \sigma_p \rangle \rangle \quad (3.1)$$

all $\sigma_i \in \mathcal{P}\langle 0, 1, \dots, n \rangle$, where $\langle 0, 1, \dots, n \rangle$ is a set of vertices and $\langle \sigma_1, \dots, \sigma_p \rangle$ is a set of simplices over those vertices. Since there are at most \aleph_0 such sets, there are at most \aleph different manifolds *homeomorphic* to the complex K .

Thus, we have the following result:

Lemma 3.3. $ZFC \vdash$ There are at most \aleph_0 homeomorphism classes of n -dimensional compact smooth manifolds. ■

See Cheeger and Kister (1970a,b), where the above reasoning is sketched.

Remark 3.4. The same result applies to the piecewise-linear class, with respect to adequate maps. ■

The differentiable case is more delicate, due to the possible existence of exotic structures (well, not in three dimensions, but we are dealing here with a more general case). For differentiable structures one may proceed as follows: if we suppose that “smooth” means “of class C^k ,” $1 \leq k \leq \infty$, a compact differentiable real n -manifold is given by a set of maps

$$f_1: B_1 \rightarrow \mathbf{R}^n, \dots, f_k: B_k \rightarrow \mathbf{R}^n, \quad k < \omega_0 \tag{3.2}$$

and

$$h_{ij}: f_j(B_i \cap B_j) \rightarrow f_i(B_i \cap B_j) \tag{3.3}$$

where

$$h_{ij} = f_i \circ f_j^{-1}|_{f_j(B_i \cap B_j)} \tag{3.4}$$

$i, j \leq k$, f_i and h_{ij} diffeomorphisms. Now the B_i can be taken to be closed balls, and the f_i and h_{ij} can be approximated (in the space of all diffeomorphisms of each B_i and $B_i \cap B_j$, with the C^k topology) by elements of a dense countable subset. Thus, the set of all systems of the form

$$p_1: B_1 \rightarrow \mathbf{R}^n, \dots, p_k: B_k \rightarrow \mathbf{R}^n \tag{3.5}$$

where $k < \omega_0$ and p_i approximates f_i in the above sense; and

$$p_{ij}: p_j(B_i \cap B_j) \rightarrow p_i(B_i \cap B_j) \tag{3.6}$$

where again p_{ij} approximates h_{ij} , is an infinite countable set. Thus, our conclusion.

We could also proceed in a less combinatorial way. We define:

Definition 3.5. If D^k is a k -dimensional disc, then $H_\lambda^n = D^\lambda \times D^{n-\lambda}$ is the *handle of index λ* .

We have the following result.

Proposition 3.6. $ZFC \vdash$ If M is a compact connected closed smooth, real n -manifold, then it is diffeomorphic to a reunion of handles $\{H_\lambda^n\}$ joined through gluing diffeomorphisms according to the following prescription:

[i] We are given a Morse function f on M whose critical points are the $\{x_0, \dots, x_\lambda, \dots, x_n\}$, and λ is the index of the critical point.

[ii] To each x_λ there corresponds a handle H_λ^n attached via a diffeomorphism f_λ .

Proof. Novikov *et al.* (1987). ■

Thus, in order to enumerate compact real, smooth n -manifolds, we proceed as follows: every compact n -manifold M admits a Morse–Smale function h such that:

- [i] $h(x_\lambda) = h(x_\mu)$ if and only if $\lambda = \mu$.
- [ii] $h(x_\lambda) < h(x_\mu)$ if and only if $\lambda < \mu$.
- [iii] There exists a single critical point x_0 and a single x_n .

For the proof see Novikov *et al.* (1987). As a consequence, the information we require in order to characterize M through a handlebody decomposition is contained in an ordered set

$$\langle x_0, \dots, \langle x_\lambda, x'_\lambda, \dots, x_\lambda^{(p)} \rangle, \dots, \langle x_\mu, \dots, x_\mu^{(q)} \rangle, \dots, x_n \rangle$$

To each critical value we then attach a “gluing” diffeomorphism $p_\lambda^{(k)}$, so that we in fact have an ordered set of ordered sets of *pairs* of the form $\langle x_\lambda^{(k)}, p_\lambda^{(k)} \rangle$, where the p ’s are taken from a countably dense set in an adequate space of diffeomorphisms (possible, since we are dealing with compact objects).

And, well, the set of all such objects is an infinite denumerable set. Therefore, we have the following result.

Lemma 3.7. ZFC \vdash There are at most \aleph_0 compact manifolds modulo diffeomorphisms.

3.2. Cylindrical Spacetime Supports in Boolean-Valued and Forcing Extensions

We have the following corollary:

Corollary 3.8. [i] ZFC \vdash There is a 1–1 and onto function $f: \omega_0 \rightarrow \mathcal{M}$, where \mathcal{M} is the set of all cylindrical spacetimes.

[ii] ZFC \vdash for all $n, n \in \omega_0$ if and only if $M_n = f(n) \in \mathcal{M}$.

We do not care about the form of f . It is enough to know that it exists. Now let $B \in \mathbf{V}$ be a complete Boolean algebra and let \mathbf{V}^B be the corresponding Boolean-valued universe. Also let $x \in \mathbf{V} \mapsto \hat{x} \in \mathbf{V}^2 \subseteq \mathbf{V}^B$ be the embedding of elements from \mathbf{V} into \mathbf{V}^B . Then we have the following result.

Proposition 3.9. $\|M_n \in \mathcal{M}\| = \bigvee_{m \in \omega_0} \|\hat{m} = n\|$.

Proof. From ZFC $\vdash \forall n((n \in \omega_0) \leftrightarrow (M_n \in \mathcal{M}))$, we get

$$\mathbf{V}^B \models \forall n((n \in \hat{\omega}_0) \leftrightarrow (M_n \in \mathcal{M}))$$

that is,

$$\|M_n \in \mathcal{M}\| = \|n \in \hat{\omega}_0\| = \bigvee_{m \in \omega_0} \|\hat{m} = n\| \quad \blacksquare$$

Corollary 3.10. $\mathbf{V}^B \models \mathcal{M} = \hat{\mathcal{M}}$.

That means: The set of all cylindrical spacetime supports gets no new elements when we expand our universe from \mathbf{V} to \mathbf{V}^B . Now let us write

$$M = \bigvee_{i \in I} (a_i \wedge \hat{M}_i), \quad \bigvee_{i \in I} a_i = 1 \quad (3.7)$$

Then we have the following.

Corollary 3.11. For M as above, $\mathbf{V}^B \models N \in \hat{\mathcal{M}}$ if and only if

$$\mathbf{V}^B \models N = M$$

So, we have “mixed” spacetimes in \mathbf{V}^B . An interesting case is given by $M = a \wedge \hat{M}_n$, $a < 1$, $a \in B$. Then we have the following.

Corollary 3.12. [i] $\|M = \emptyset\| = a^*$. [ii] If we put $M' = (a \wedge \hat{M}_n) \vee (a^* \wedge \emptyset)$, then $\mathbf{V}^B \models M = M'$. [iii] $\mathbf{V}^B \models M \in \hat{\mathcal{M}}$.

Corollary 3.13. $\mathbf{V}^B \models (M = \emptyset) \vee (M = \hat{M}_n)$.

Now, as B can be seen as the underlying algebra of a strictly positive probability measure μ on an adequate space, the preceding result means that there is a $\mu(a^*)$ chance that M “is” the empty set, or there is a $\mu(a)$ chance that M “is” a standard manifold.

Finally let us cut down from \mathbf{V} to a countable transitive model \mathbf{M} for ZFC, and let \mathbf{M}^B be the corresponding Boolean extension. We then go to the quotient as described in Section 2 and get the forcing extension $\mathbf{M}(B)$. The situation in $\mathbf{M}(B)$ is somewhat different, as we have no mixtures:

Corollary 3.14. $\mathbf{M}(B) \models$ Every $M \in \mathcal{M}$ is standard.

Now for M' as above, we have:

Corollary 3.15. Either $\mathbf{M}(B) \models M' = \hat{M}_n$ or $\mathbf{M}(B) \models M' = \emptyset$. \blacksquare

Thus, things are much neater in forcing extensions, as it concerns cylindrical supports for spacetimes: everything is either standard or nothing. We will require those results in Section 5.

Notice that the enumeration techniques used here to determine the cardinality of equivalence classes of smooth manifolds modulo diffeomorphisms both in the compact and in the noncompact case allow us to code each cylindrical spacetime by a finite sequence of symbols, while noncompact spacetimes are coded by denumerably infinite sequences of symbols.

Obviously, infinitely many sequences may code elements of the same diffeomorphism class, but we will easily notice (and make explicit in the next section) that set-theoretic genericity appears for noncompact spaces in the way one pastes together local coordinate domains. In that sense, it is a global property of those manifolds.

4. GENERIC NONCOMPACT SPACETIME SUPPORTS

We first quote a well-known result: let M be a noncompact C^∞ real 4-manifold. Then, the following holds.

Proposition 4.1. $ZFC \vdash M$ admits a nondegenerate Lorentzian metric tensor.

Proof. See Steenrod (1951). ■

4.1. The Set of Noncompact Spacetime Supports Has the Power of the Continuum

In contrast to Proposition 3.1, we have the following.

Proposition 4.2. $ZFC \vdash$ There are 2^{\aleph_0} diffeomorphism classes of noncompact differentiable real C^k n -manifolds, $1 \leq k \leq \infty$, $n > 1$.

Proof. We first show that there are at least 2^{\aleph_0} diffeomorphism classes: let ω_0 be given; associate to each $n \in \omega_0$ a 2-torus $T^2 = S^1 \times S^1$. We index all those (evidently diffeomorphic) tori by that positive integer, and write $T^2(n)$ for each one of those tori. Now form the connected sum

$$T^2(0) \# \cdots \# T^2(n-1) \# T^2(n) \# T^2(n+1) \# \cdots \quad (4.1)$$

We get a linear chain of tori,

$$\# \sum_{i \in \omega_0} T^2(i) \quad (4.2)$$

where $\# \sum$ denotes the connected sum over the corresponding indices. Clearly, $\# \sum_{i \in \omega_0} T^2(i)$ can be made into a smooth noncompact real 2-manifold.

In order to establish a correspondence between some noncompact 2-manifolds and a set of cardinality equal to 2^{\aleph_0} , we proceed as follows: if $2 = \{0, 1\}$, we form the product set 2^{ω_0} . To each sequence $\alpha \in 2^{\omega_0}$, we form out of $\# \sum_{i \in \omega_0} T^2(i)$ a new smooth manifold by the following rules:

1. If $\alpha(n) = 0$, do nothing.
2. If $\alpha(n) = 1$, $\#$ -sum to $T^2(n) \subset \# \sum T^2(i)$ another torus T^2 , so that the manifold chain $\# \sum T^2(i)$ will have a “dangling ring” at n .

If we denote by $M(\alpha)$ the new manifold we just built, we check that for $\alpha, \beta \in 2^{\omega_0}$, $\alpha \neq \beta$ if and only if the corresponding manifolds $M(\alpha) \not\cong M(\beta)$, “ \cong ” meaning “not diffeomorphic to.” Thus, the set $\{M(\alpha): \alpha \in 2^{\omega_0}\}$ has the power of the continuum, and the n -manifolds $\{\mathbb{R}^{n-2} \times M(\alpha): \alpha \in 2^{\omega_0}\}$ form a set of nondiffeomorphic noncompact n -manifolds with the power of the continuum.

We then show that there are at most 2^{\aleph_0} diffeomorphism classes of noncompact n -manifolds, $n > 1$. Every differentiable manifold has a locally finite atlas. Thus, every differentiable manifold can be represented by a locally finite atlas plus its (countable) set of transition functions. The set of all such objects, adequately coded as a denumerably infinite sequence of letters, some of which run over denumerable sets and some other (the local maps and the transition functions) which run over continuum-many objects, can have at most the power of the continuum.

Thus, we conclude our argument. ■

Corollary 4.3. ZFC \vdash There exists a 1-1 and onto function $f: 2^{\aleph_0} \rightarrow \mathcal{N}$, where \mathcal{N} is the set of all noncompact 4-dimensional manifolds that support spacetimes modulo diffeomorphisms.

4.2. Generic Spacetime Supports

Now, if we take $V = L$ and if we put $B = RO(2^{\omega_0 \times \omega_1})$, the regular open algebra of the product space $2^{\omega_0 \times \omega_1}$, then we have the following result.

Lemma 4.4. $L^B \models$ There are 2^{\aleph_0} set-theoretically generic noncompact spacetime supports modulo diffeomorphisms.

Proof. First we notice that $L^B \models$ “There are 2^{\aleph_0} set-theoretically generic subsets of ω_0 ” (Bell, 1985). From the preceding corollary we get our result. ■

We thus conclude that, from the strictly geometric point of view, there are a great many (actually a whole continuum of) set-theoretically generic support manifolds for spacetimes in adequate set-theoretic models. The questions that arise at the present point are:

First, are those spacetime supports physically meaningful? What is their physical import? How do we *physically* (that means, with respect to physically sensible theoretic constructs) recognize set-theoretic genericity in a spacetime support manifold?

Second, how does set-theoretic genericity behave with respect to diffeomorphisms? That is to say, which is its behavior with respect to general relativistic covariance?

Third, how can we detect set-theoretic genericity in the world around us? Can we *experimentally* detect it at all?

We deal with those questions in the following sections.

5. GENERAL COVARIANCE AND SPACETIME SUPPORTS

A spacetime $\langle M, g \rangle$ is supported by a separable Hausdorff manifold M . In a ZFC universe \mathbf{V} , let us be given a countable open cover for M , $\mathcal{X} = \{U_i : i \in \omega_0\}$, so that:

- [i] $\bigcup_{i \in \omega_0} U_i = M$.
- [ii] For $i \neq j$, it is false that $U_i \subseteq U_j$ or $U_j \subseteq U_i$.

Then, we have the following.

Lemma 5.1. ZFC \vdash Let $f \in 2^{\omega_0}$, and to each such f associate a subset $\mathcal{Y}_f \subset \mathcal{X}$ given by

$$U_i \in \mathcal{Y}_f \leftrightarrow f(i) = 1 \tag{5.1}$$

$$U_i \notin \mathcal{Y}_f \leftrightarrow f(i) = 0 \tag{5.2}$$

Then [i] the set

$$\bigcup_{v \in \mathcal{Y}_f} V$$

is open, and [ii] for $f \neq f'$,

$$\bigcup_{v \in \mathcal{Y}_f} V \neq \bigcup_{w \in \mathcal{Y}_{f'}} W$$

f and $f' \in 2^{\omega_0}$.

Proof. Immediate; [ii] in particular is a consequence of the same-numbered condition above. ■

Let $B = RO(2^{\aleph_0}) \in \mathbf{V}$, the regular open algebra of the product space 2^{\aleph_0} . Then, we have the following result.

Proposition 5.2. $\mathbf{V}^B \models$ There is an open set $U \subset \hat{M}$ so that for all standard open sets $\hat{V} \subset \hat{M}$, $U \neq \hat{V}$.

Proof. $\mathbf{V}^B \models$ “ $(\hat{2}^{\omega_0})$ is a proper subset of $2^{\hat{\omega}_0}$.” We then pick up $f \in \mathbf{V}^B$, so that $\mathbf{V}^B \models$ “ $f \in 2^{\hat{\omega}_0}$ and for all standard $\hat{g} \in 2^{\hat{\omega}_0}$, $f \neq \hat{g}$.”

Then, $\mathbf{V}^B \models$ “For all $\hat{g} \in 2^{\hat{\omega}_0}$,

$$\bigcup_{v \in \mathcal{Y}_f} V \neq \bigcup_{w \in \mathcal{Y}_{\hat{g}}} W” \quad \blacksquare$$

So, there are new generic open sets in M 's topology in the Boolean extension \mathbf{V}^B [and also in the corresponding forcing extension $\mathbf{M}(B)$]. But are such new open sets *physically* new? In general relativity, Einstein's “principle of general covariance” means that all objects are defined modulo

diffeomorphisms (local or global). As we will see, the situation is not a simple one, as we juggle with Boolean and forcing extensions. We will try to unravel its main features through a few results and examples.

In what follows, “open ball” means an open set which is diffeomorphic to \mathbf{R}^4 with its usual differentiable structure.

Proposition 5.3. $\mathbf{V}^B \models$ Every open ball $U \subset M$ is diffeomorphic to a set-theoretically standard open ball $\hat{W} \subset M$.

Proof. Immediate, since all such open balls are diffeomorphic in M . ■

Proposition 5.4. $\mathbf{V}^B \models$ If $K \subset M$ is compact such that its interior is a smooth submanifold of M and such that its boundary is also smooth, then K is diffeomorphic to a standard compact $\hat{V} \subset M$ with a smooth interior and a smooth boundary.

Proof. Along the lines of the proof of Proposition 3.1. ■

We now give a few examples to show explicitly how some generic open sets in M (within a universe \mathbf{V}^B) can be diffeomorphic to standard open sets. We proceed in the forcing extension $\mathbf{M}(B)$ associated to \mathbf{M}^B rather than in the Boolean extension, since in that case the argument is much more intuitive:

Example 5.5. In $\mathbf{M}(B)$, let the real line \mathbf{R} be covered with a countable collection of open intervals centered at each $x \in \mathbf{Z}$ with diameter equal to $1 + \varepsilon$, where $0 < \varepsilon \ll 1$. We then have a cover of “slightly” overlapping sets. If \mathcal{X} is such a cover, any $\bigcup V_i \subset \mathbf{R}$, $V_i \in \mathcal{Y}$ and \mathcal{Y} properly included in \mathcal{X} , will be open and disconnected. Fix an enumeration for the sets in the cover \mathcal{X} , and let $u \subset \hat{\omega}_0$ be a set-theoretically generic set. The open set $X_u = \bigcup V_i, i \in u$, has \aleph_0 connected components (\aleph_0 “pieces”). However, such a set is immediately seen to be diffeomorphic to, say, $Z = \bigcup V_i, i \text{ even}$.

Example 5.6. Again in $\mathbf{M}(B)$, let us cover \mathbf{R}^2 with a countable set of open square “tiles” centered at each $x \in \mathbf{Z}^2$ and with sides equal to $1 + \varepsilon$ aligned to the axes of a rectangular coordinate system. We restrict our attention to the first quadrant. We enumerate elements of the cover according to Cantor’s well-known rule for the enumeration of rationals, $(0; 0), (0; 1), (1; 0), (2; 0), (1; 1), (0; 2), (0; 3), \dots$, so that we may connect those points with a continuous line. Given that enumeration, for a generic $u \in \hat{\omega}_0$, we take out the tiles whose indices fall in u . We then get a plane with \aleph_0 holes at the x ’s in u . However, since we can draw a continuous non-self-crossing line through all positive pairs of integral coordinates in \mathbf{R}^2 , we can “straighten up” that line, so that the holes that we have punctured in \mathbf{R}^2 will fall nicely, say, in the first column in the positive quadrant. Again we have a generic open set (the plane with holes at the places coded by u) which is diffeomorphic to a standard open set in \mathbf{R}^2 .

Remark 5.7. That example easily generalizes to the case when we puncture holes all over the plane according to some generic set coding; it is also valid for any \mathbf{R}^n , $n \in \omega_0$.

Remark 5.8. Physically \mathbf{R}^4 less the punctured holes can be taken to represent a flat spacetime manifold with \aleph_0 test particles in it.

We have seen that set-theoretically generic open sets in M can be diffeomorphic to standard open sets. We now ask a related question: given a standard $\hat{M} \in \mathbf{M}(B)$, do we have a set-theoretically generic open submanifold $X \subset \hat{M}$ that *cannot* be diffeomorphically mapped on a standard manifold? The answer is given in the next example.

Example 5.9. Let us be given the one-sided infinite chain of tori $\# \sum_i T^2(i)$ given in Proposition 4.2 within $\mathbf{M}(B)$. Clearly, that manifold is a standard manifold. Given the characteristic function f_u for a generic $u \in \hat{\omega}_0$, cut out a disc from each $T^2(n) \subset \# \sum_i T^2(i)$ if and only if $f_u(n) = 1$, and do nothing otherwise. The new noncompact submanifold we thus obtain is open and clearly generic and cannot be made diffeomorphic to a standard manifold.

Remark 5.10. However, we should notice that set-theoretic genericity is not necessarily determined by the spacetime homeomorphism class, since the existence of continuum-many exotic \mathbf{R}^4 's implies—within an adequate set-theoretic model—the existence of continuum-many set-theoretically generic exotic \mathbf{R}^4 's that are homeomorphic to the standard \mathbf{R}^4 . We also note that the Taubes end-construction that led to the proof of the existence of continuum-many exotic copies of the \mathbf{R}^4 is mirrored (in a much simpler way) in the idea behind the previous example (Taubes, 1987).

6. GENERICITY, RANDOMNESS, AND AN UNDECIDABLE QUESTION

We can talk sensibly about the set \mathcal{S} of all spacetime supports within ZFC, since all n -dimensional, real, smooth manifolds are diffeomorphic to embedded submanifolds of \mathbf{R}^{2n+1} . As a result, we can realize diffeomorphic images of any spacetime within \mathbf{R}^9 ; thus, $\mathcal{S} \subset^{\text{diff}} \mathcal{P}(\mathbf{R}^9)$. However, at first no spontaneous topological structure for \mathcal{S} seems to be available (what do we mean when we try to figure out how a sequence of spacetime supports “approximates” a given spacetime support?)

Yet there are several quite reasonable topologies for \mathcal{S} . We describe here one that best suits our purposes; it is a rather simple topological structure for the set of all 4-dimensional, smooth, real, noncompact manifolds (since we exclude compact 4-manifolds from the class of all spacetimes); in the topology we propose \mathcal{S} has a quite well-behaved geometry,

since it is a Polish space; the idea is that manifolds in \mathcal{S} are “close” whenever they share a sufficiently long “initial segment” sequence, in a manner to be specified below.

We use that topology in order to show that a tentative identification of set-theoretically generic spacetime supports with random spacetime supports leads to an undecidable question within ZFC. The suggestion that generic set-theoretic objects are somehow random objects goes back to the Solovay–Scott theory of Boolean-valued models and seems to lie behind the motivation for those models. Scott (1967) explicitly constructs his model for the real line out of random functions on the reals, and Solovay’s work on the Lebesgue measure problem has led to at least one definition for randomness in sequence spaces that was shown to be equivalent to the Kolmogorov–Chaitin–Martin–Løf characterization (Solovay, 1970; Chaitin, 1987).

However, as we show here, set-theoretic genericity seems to go beyond mere randomness; this is the content of the simple undecidability result that concludes the present section.

6.1. A Topology for \mathcal{S}

We need the following results; here “smooth” means “ C^∞ .”

Proposition 6.1. ZFC \vdash If M is a real, smooth n -manifold, then M is triangulizable.

Proof. See Whitehead (1940) and Cairns (1940). ■

We are dealing with spacetime supports; that means that our interest is restricted to 4-dimensional, real, smooth noncompact manifolds. Thus, we require:

Proposition 6.2. ZFC \vdash If M is a noncompact 4-dimensional real topological manifold, then it is smoothable, that is, it can be given a smooth atlas which is compatible with the topological manifold structure.

Proof. See Quinn (1982). ■

Therefore, we have the following result.

Corollary 6.3. ZFC \vdash If M is a simplicial complex with a 4-dimensional real topological manifold structure, then M is smoothable.

We also need some other properties of the atlases for our manifolds; they are listed below. Given an atlas for an n -manifold M where the local coordinate domains are the $\{B_i : i \in \omega_0\}$, and $\bigcup_i B_i = M$, we ask that:

- [1] Each coordinate domain B_i , $i \in \omega_0$, in the atlas for the n -manifold M should be diffeomorphic to an open n -ball.

- [ii] The atlas satisfies the finite intersection property, that is, given a coordinate domain B_i , it will meet at most a finite number of other coordinate domains B_j in the atlas.
- [iii] For any two $i, j, i \neq j, B_i \not\subseteq B_j$ or $B_j \not\subseteq B_i$.

For those atlases, any n -dimensional smooth manifold is given by two (finite or denumerably infinite) sequences of maps

$$p_1: B_1 \rightarrow \mathbf{R}^n, \dots, p_i: B_i \rightarrow \mathbf{R}^n \dots \quad (6.1)$$

$$\dots p_{ij}: p_j(B_i \cap B_j) \rightarrow p_i(B_i \cap B_j) \dots \quad (6.2)$$

where the p_i and p_{ij} are the local coordinate-defining diffeomorphisms, $p_{ij} = p_i \circ p_j^{-1}|_{p_j(B_i \cap B_j)}$.

We abbreviate:

$$B_{ijk\dots m} = B_i \cap B_j \cap \dots \cap B_m \quad (6.3)$$

Then, we have the following.

Definition 6.4. If M is an n -dimensional, smooth, real manifold, then the cover $\{B_i\}$ is a *good cover* if and only if all $B_{ijk\dots m}$ are diffeomorphic to an open n -ball.

Proposition 6.5. ZFC \vdash Given M as above, M has a good cover which satisfies conditions [i]–[iii]. Moreover, that cover is homeomorphic to the open stars in a triangulation for M .

Proof. Proof is given in Bott and Tu (1982), pp. 42 and 190. There it is shown that, given a cover that satisfies conditions [ii] and [iii] at the top of this subsection, we can always choose a good cover which refines it. ■

As we wish to code smooth manifolds as finite or denumerably infinite sequences of integers, we prepare an infinite effective enumeration of all possible sets $\{B_{ij\dots k}\}$:

We require:

[*] Given any $B_{ij\dots m}$, then $i < j < \dots < m$.

[**] We order the $B_{ij\dots m}$ by the following rules: (1) We first order as a growing sequence the sums $s = i + j + \dots + m$; (2) given each value s_0 for that sum, we devise an ordering rule for the finite collection of sets $B_{ij\dots m}$ so that $i + j + \dots + m = s_0$.

We get something like

$$\mathcal{C} = \{B_{01}, B_{02}, B_{03}, B_{12}, B_{012}, B_{01}, B_{13}, B_{013}, \dots\} \quad (6.4)$$

We have shown the following.

Lemma 6.6. $ZFC \vdash$ There is an effective (that is, computable) 1-1 and onto map $\lambda: \mathcal{C} \rightarrow \omega_0$. The map λ also induces an effective 1-1 and onto map from the set $\text{Fin}(\mathcal{C})$ of all finite subsets of \mathcal{C} , $\lambda^*: \text{Fin}(\mathcal{C}) \rightarrow \omega_0$.

We denote by $(\omega_0)_{\mathcal{C}}$ the coding we get for all finite subsets of \mathcal{C} onto the natural numbers. Similar codings denoted $(\omega_0)_{\mathcal{C}_i}$ will be used below for some $\mathcal{C}_i \subset \mathcal{C}$.

We will use the effectiveness stated by the preceding lemma in the next subsection. The $\mathcal{C}_i \subset \mathcal{C}$ are also effectively constructed. Now in order to map \mathcal{S} one-one and onto $(\omega_0)^{\omega_0}$:

1. We map all coordinate domains in our good cover $\mathcal{B} = \{B_i: i \in \omega_0\}$ 1-1 and onto ω_0 ; let $(\omega_0)_{\mathcal{B}}$ denote the positive integers under such a coding.

2. We select a dense countable subset $\mathcal{H} = \{p_i: i \in \omega_0\}$ in the space of all diffeomorphisms from the B_i onto \mathbf{R}^4 ; we then form \mathcal{H}^{ω_0} . A coordination for the cover \mathcal{B} is then a couple $\langle \mathcal{B}, \varphi \rangle$, $\varphi \in \mathcal{H}^{\omega_0}$, given by the sequences $\langle n, \varphi(n) \rangle$, $n \in (\omega_0)_{\mathcal{B}}$.

3. We then describe the good cover through its elements in \mathcal{C} . We start with B_0 . B_0 meets a finite number of elements in the cover \mathcal{B} , that is, a finite set $C_0 \subset \mathcal{C}$, where all elements in C_0 are of the form $B_{0ij\dots m}$. As C_0 is a finite subset, given the coding described above for $(\omega_0)_{\mathcal{C}_0}$, where $\mathcal{C}_0 = \{\dots B_{0ijk\dots m} \dots\}$, we determine a positive integer n_0 . We finally put $F_M(0) = n_0$.

4. We choose the smallest index after 0 in the $B_{0ij\dots}$, say j , and put $j = 1$. We then renumber everything accordingly (it is just a permutation), and proceed to the next step, with the new B_1 .

5. Given constructions for all B_i , $i < k$, we enter the k th step. We exclude from \mathcal{C} all sets that have already been used in the preceding steps and are not available any longer; we then get \mathcal{C}_k and $(\omega_0)_{\mathcal{C}_k}$. We choose the portion of the good cover with index k and get the corresponding n_k . We write $F_M(k) = n_k$. We then choose the smallest i in the $B_{kij\dots}$ and renumber it $i = k + 1$; and proceed to the next step.

Remark 6.7. Notice that we have described a map from the cover for M into ω_0 . However, since any $F \in (\omega_0)^{\omega_0}$ will be thus associated to a good cover for a manifold in \mathcal{S} , that is, we can have any $F = F_M$ in the preceding construction, each element of \mathcal{S} is denoted by a unique sequence $\langle F_M(n), \varphi(n) \rangle$, whence it is easy to get the desired 1-1 and onto map $\iota: \mathcal{S} \rightarrow (\omega_0)^{\omega_0}$.

We have proved the following.

Proposition 6.8. $ZFC \vdash$ There is a 1-1 and onto map $\iota: \mathcal{S} \rightarrow (\omega_0)^{\omega_0}$.

Corollary 6.9. $ZFC \vdash$ There is a 1-1 and onto correspondence $\kappa: \text{Ir} \rightarrow \mathcal{S}$, between the set of all irrationals $\text{Ir} \subset \mathbf{R}$ and \mathcal{S} .

Proof. See Hinman (1978). ■

Corollary 6.10. ZFC \vdash There is a 1-1 correspondence $\rho: \text{BIr} \rightarrow \mathcal{S}$ between the set of all binary irrationals $\text{BIr} \subset [0, 1] \subset \mathbf{R}$ and \mathcal{S} .

Proof. Binary irrationals in the unit interval are the reals whose binary expansion does not end in an infinite succession of zeros or ones. Proof is as above. ■

Definition 6.11. We endow \mathcal{S} with the topology induced (and coinduced) from ι , κ , or ρ above. \mathcal{S} is thus a complete and separable metric space, that is, a *Polish space*; moreover, it is totally disconnected.

Remark 6.12. The above construction looks quite cumbersome; it could be abbreviated as follows: from Corollary 6.3, we could realize all infinite complexes with a topological manifold structure and dimension 4 within \mathbf{R}^{ω_0} with the product topology, which is also a Polish space. If we now restrict our attention to complexes thus realized within \mathbf{R}^{ω_0} whose vertices have rational components, we get a residual $G_\delta \subset \mathbf{R}^{\omega_0}$ which again is a Polish space. Moreover, there is a homeomorphism between that G_δ and \mathcal{S} .

However, our construction emphasizes what is effective and what is not in the elements of \mathcal{S} , while effectiveness is much less clear in the result we have just sketched.

6.2. Computability, Randomness, and Undecidability in \mathcal{S}

We begin as follows.

Definition 6.13. $M \in \mathcal{S}$ is *computable* if and only if F_M is computable.

Then, we have the following.

Proposition 6.14. ZFC \vdash The map ι sends computable objects over computable objects.

We can finally assert the following.

Definition 6.15. $M \in \mathcal{S}$ is *K[olmogorov]-C[haitin]-random* if $M = \rho(\sigma)$, and $\sigma \in \text{BIr}$ is an infinite one-sided KC-random binary sequence.

For that characterization for randomness see Chaitin (1987). Then the following holds.

Proposition 6.16. ZFC \vdash The set of all KC-random spacetime supports in \mathcal{S} is residual and of full measure, in the induced topology and measure.

Proof. Through the homeomorphism ρ . ■

Can we equate “randomness” and “set-theoretic genericity”? Set-theoretic genericity, as (intuitively) “opposed” to Gödel’s constructivity,

seems to introduce faceless, irreducibly random objects in our theory. That intuition is reinforced by the Scott–Solovay Boolean models, which originally were built out of random functions and Borel algebras. However, we have the following.

Proposition 6.17. The sentence “The set of KC-random physically distinct spacetime supports equals the set of set-theoretically generic physically distinct spacetime supports modulo a meager set” is undecidable within ZFC.

We must first say what we mean by “physically distinct” spacetimes. We factor \mathcal{S} by the action of the group of smooth diffeomorphisms; we proceed as follows—after realizing the elements of \mathcal{S} within \mathbf{R}^9 through Whitney embeddings, we collapse diffeomorphic spacetimes modulo the action of the group \mathcal{D} of smooth diffeomorphisms of \mathbf{R}^9 . We then get the quotient space \mathcal{S}/\mathcal{D} , which we endow with the projection topology.

Definition 6.18. $\mathcal{N} = \mathcal{S}/\mathcal{D}$ is the set of *physically distinct* spacetime supports.

$\mathcal{N} = \mathcal{S}/\mathcal{D}$ is again a Polish space, and the set of KC-random objects in \mathcal{S} is again mapped onto a residual set.

We have the following.

Lemma 6.19. $\mathbf{V} = \mathbf{L} \models \neg$ “The set of KC-random physically distinct spacetime supports equals the set of set-theoretically generic physically distinct spacetime supports modulo a meager set.”

Proof. Immediate, since $\mathbf{V} = \mathbf{L} \models$ “The set of generic objects is empty.” ■

For the affirmative result: we start from a universe $\mathbf{L}^B \models \text{ZFC} + 2^{\aleph_0} > \aleph_1 + \text{Martin's Axiom}$. If $|X|$ denotes X 's cardinality, we know the following.

Proposition 6.20. $\mathbf{L}^B \models$ “If X is Polish and $U \subset X$ is such that $|U| < 2^{\aleph_0}$, then U is meager in X .”

Proof. See Kunen (1983). ■

Also, we have the following.

Lemma 6.21. $\mathbf{L}^B \models “|\hat{\mathcal{S}}| = \aleph_1.”$

Proof. By cardinal conservation (Bell, 1985). ■

Lemma 6.22. $\mathbf{L}^B \models$ “The set of generic spacetime supports $\mathcal{S} - \hat{\mathcal{S}}$ is residual in \mathcal{S} .”

Thus, as the residual sets on any topological space X form a filter, we have the following result.

Proposition 6.23. $L^B \models$ “The set of KC-random physically distinct spacetime supports equals the set of set-theoretically generic physically distinct spacetime supports modulo a meager set.”

Proof. Immediate, by going to the quotient in \mathcal{S}/\mathcal{D} . ■

Remark 6.24. Do they coincide? The answer is, no. First, there are Gödel-constructible random sequences in the $\text{BIR} \in L^B$. Also, if $u \in \text{BIR} \in L^B$ is generic, we can easily get another sequence $f(u)$ which does not satisfy, say, the law of large numbers, and which will not be constructible.

Thus, we have seen that we cannot *prove* within ZFC that set-theoretically generic objects equal KC-random objects (modulo some irrelevant set). Now can we somehow sensibly and rigorously *argue* that both sets coincide modulo some small set? We can.

Remark 6.25. Suppose that we believe that there is a proper noncountable class $V \models \text{ZFC}$. We then use the informal Löwenheim–Skolem theorem to cut down from V to a countable model M , which we further restrict to a constructible countable universe $L \subset V$, everything being made transitive through Mostowski collapsing, if needed. Then we have that the binary irrationals $(\text{BIR})^L \subset \text{BIR}$ in V , due to transitivity. When we make a forcing extension $L(g)$, we require a generic filter g , which in our case can be taken in 2^ω . Now that filter $g \notin L$, but will lie in $\text{BIR} - (\text{BIR})^L$. We can easily show that the set (with respect to V) of all generic g thus obtained is residual and has full measure in BIR . Moreover, it *does not* coincide with the set of all KC-random binary sequences in BIR , for there are random binary sequences in BIR that are not set-theoretically generic through the usual construction of g (those in L), while there are generic filters g that are not random (they violate, say, the law of large numbers) (Feferman, 1965; Hinman, 1978; Doria *et al.*, 1987).

Thus, we can *informally* argue that “almost all” generic sequences are random—and therefore that randomness and genericity will informally coincide for most spacetime supports. However, we cannot prove that within ZFC.

7. COMMENTS AND INTERPRETATIONS

We can thus summarize our results in the previous sections:

1. Cylindrical spacetime supports from a countable set, modulo diffeomorphisms. Thus, we gain nothing when we make forcing extensions in the underlying set-theoretic model. Yet, one should pay attention to possible physically sensible interpretations for those spacetime supports that are in part equal to a given manifold, and that are in part equal to the

empty set, within a Boolean extension. We believe that they can be given a quite interesting fractal “dust” structure (da Costa and Doria, 1989a).

2. Set-theoretically generic objects appear when we consider the class of arbitrary noncompact 4-manifolds in adequate forcing extensions. Depending on the model, set-theoretic genericity may even equal topological genericity (modulo a meager set). Thus, set-theoretically generic spacetime supports are the typical spacetime supports in such models.

3. Set-theoretic genericity for spacetime supports is a *global* phenomenon. Also, it is *not* determined by the manifold’s homeomorphism class, nor by its connectivity (or cohomology or homotopy) properties, despite the fact that such properties may sometimes imply a generic structure for the manifold. Local domains diffeomorphic to an open ball and compact domains are always standard.

4. Finally, we can intuitively suggest that “most” set-theoretically generic objects should look like random objects. One must be careful here, since when we try to prove it within ZFC, a well-known argument shows that the identification between genericity and randomness is an undecidable question. Yet we can still argue in favor of that identity with the help of an informal but mathematically sensible proof of the coincidence of both sets of objects in the case in which we are interested, modulo a meager set.

We have not explored here the case of generic *fields* over standard domains. Those sets of fields have in most cases the power of the continuum, and even if we factor them modulo diffeomorphisms, we still get a set with the power of the continuum. That is the case, for instance, of gravitational metric tensors over a fixed spacetime support. We intend to discuss that situation elsewhere, since it has been suggested that topologically generic metric tensors on a given spacetime are random (Fischer, 1970).

We thus propose that the search for set-theoretic genericity in the physical world around us should be a search for randomness in that world. We may suggest the following scenario: given an “almost standard” forcing extension for the ZFC axioms (an extension where the whole of classical mathematics is true), we believe that set-theoretic genericity might appear as some sort of totally uncontrolled or unexpected randomness within our observational data. Somehow our theoretical constructs and observational tools should be able to cope with some, say, well-behaved randomness within our experience. Yet there might be some other irreducible kind of randomness within that empirical domain that would best be dealt with by the concept of set-theoretic genericity.

The main problem would then be how to separate the “nice” kind of randomness from the “nasty” species—if our suggestion follows the right track in those matters.

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