

# Decision Making for Inconsistent Expert Judgments Using Signed Probabilities

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**Abstract.** In this paper we provide a simple random-variable example of inconsistent information, and analyze it using a system of signed probabilities inspired by quantum mechanics.

## 1 Introduction

In recent years the quantum-mechanical formalism has been used to model economic and decision-making processes (see [1,2] and references therein). The success of such models may originate from several related issues. First, the quantum formalism leads to a propositional structure that does not conform to classical logic [3]. Second, that quantum observables do not satisfy Kolmogorov's axioms of probability [4]. Third, that quantum mechanics describes experimental outcomes that are highly contextual [5,6,7,8,9]. Such issues are connected because the logic of quantum mechanics, represented by a quantum lattice structure [10], leads to upper probability distributions and thus to non-Kolmogorovian measures [11,12,13], while contextuality leads to the nonexistence of a joint probability distribution [14,15,16].

Both from a foundational and from a practical point of view, it is important to ask which aspects of quantum mechanics are actually needed for social science models. For instance, the Hilbert space formalism leads to non-standard logic and probabilities, but the converse is not true: one cannot derive the Hilbert space formalism solely from weaker axioms of probabilities or from quantum lattices. Furthermore, the quantum mathematical structure yields non-trivial results such as the impossibility of superluminal signaling with entangled states [17]. This type of results are not necessary for a theory of social phenomena [16], and we should ask what are the minimalistic mathematical structures suggested by quantum mechanics that reproduce the relevant features of quantum-like behavior.

In a previous article, we used reasonable neurophysiological assumptions to create a neural-oscillator model of behavioral Stimulus-Response theory [18]. We then showed how to use such model to reproduce quantum-like behavior [19]. Finally, in a subsequent article, we remarked that the same neural-oscillator model could be used to represent a set of observables that could not correspond to quantum mechanical observables [20]. These results suggest that one of the main quantum features relevant to social modeling is contextuality, represented

by a non-Kolmogorovian probability measure, and that imposing a quantum formalism may be too limiting.

In this paper we try to provide a simple yet realizable case where quantum decision-making models fail to properly describe. We start by showing an example of such case, and then proving that it is not possible to model it by using quantum interference. We then propose the use of non-standard measures to represent certain decision-making problems as a possible move to incorporate the main features present in the quantum formalism. We organize this paper in the following way. First, we start with our simple model where expert judgments lead to inconsistencies (see also [20]). Then, we approach this problem with standard Bayesian probabilistic methods. The complexity and arbitrariness of such methods motivate the use of non-standard statistics as an alternative. However, instead of modifying Kolmogorov's conjunction axiom, we suggest the use of signed (or negative) probability distributions. We then show that the example considered is significantly more treatable with signed probabilities than with standard approaches. We end with some comments about signed probabilities and their interpretations and possible uses.

## 2 Inconsistent Information

As mentioned, the use of the quantum formalism in the social sciences originates from the observation that Kolmogorov's axioms are violated in many situations [1,2]. Such violations in decision-making seem to indicate a departure from a rational view, and in particular to thought-processes that may involve irrational or contradictory reasoning, as is the case in non-monotonic reasoning. Thus, when dealing with quantum-like social phenomena, we are frequently dealing with some type of inconsistent information, usually arrived at as the end result of some non-classical (or incorrect, to some) reasoning. In this section we examine the case where inconsistency is present from the beginning.

Though in everyday life inconsistent information abounds, standard classical logic has difficulties dealing with it. For instance, it is a well know fact that if we start with two contradictory propositions,  $A$  and  $\neg A$ , then the logic becomes trivial, in the sense that all formulas in such logic are theorems (i.e., if  $B$  is a formula, then  $\neg B$  is also a formula, and both are theorems). To deal with such difficulty, logicians have proposed modified logical systems, such as paraconsistent logics [21]. Here, we will discuss how to deal with inconsistencies not from a logical point of view, but instead from a probabilistic one.

Inconsistencies of expert judgments are often represented in the probability literature by measures corresponding to the experts' subjective beliefs [22]. It is often argued that this subjective nature is necessary, as each expert makes statements about outcomes that are, in principle, available to all experts, and disagreements come not from sampling a certain probability space, but from personal beliefs. For example, let us assume that two experts, Alice and Bob, are examining whether to recommend the purchase of stocks in company  $X$ , and each gives different recommendations. Such differences do not necessarily emerge

only from the objective data available to both experts, but also from their own interpretations of it (e.g, which information is relevant, what is the meaning of different trends, etc). In some cases the inconsistencies are evident, as when Alice gives advice at odds with Bob (say, Alice recommends buy, and Bob recommends sell), and a decision maker would have to reconcile those differences<sup>1</sup>.

The above example provides a simple case of inconsistency. A more subtle case is when the totality of experts have inconsistent beliefs but subgroups seem to be consistent. For example, each expert, with a limited access to information, may form, based on different contexts, fully formed and locally consistent beliefs, without directly contradicting other experts (local level), but when we take the totality of the information provided by all of them (global level) and try to arrive at possible inferences we arrive at contradictions. Here we want to create a simple random-variable model that incorporates expert judgments that are locally consistent but globally inconsistent. This model, inspired quantum entanglement, will be used to show the main features of signed probabilities as applied to decision making.

Let us start with three  $\pm 1$ -valued random variables,  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$ , with zero expectation. It is a well-known result that if such random variables have correlations that are too strong then there is no joint probability distribution [24], and therefore there can be no way to assign values to such variables that are consistent with such correlations. To understand this, imagine the following extreme case, representing the strongest possible negative correlations between  $\pm 1$ -valued random variables:  $E(\mathbf{XY}) = E(\mathbf{YZ}) = E(\mathbf{XZ}) = -1$ . Imagine that in a given trial we draw  $\mathbf{X} = 1$ . From  $E(\mathbf{XY}) = -1$  it follows that  $\mathbf{Y} = -1$ , and from  $E(\mathbf{YZ}) = -1$  that  $\mathbf{Z} = 1$ . But this is in clear contradiction with  $E(\mathbf{XZ}) = -1$ , which requires  $\mathbf{Z} = -1$ . Of course, the problem is not that there is a mathematical inconsistency, but that it is not possible to find a probabilistic sample space for which the variables  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  have such strong correlations. Another way to think about this is that the the  $\mathbf{X}$  measured together with  $\mathbf{Y}$  is not the same one as the  $\mathbf{X}$  measured with  $\mathbf{Z}$ , i.e., the values of  $\mathbf{X}$  depend on the context.

The above example posits a deterministic relationship between all random variables, but the inconsistencies persist even when weaker correlations exist. In fact, Suppes and Zanotti [24] proved that a joint probability distribution for  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  exists if and only if

$$\begin{aligned} -1 &\leq E(\mathbf{XY}) + E(\mathbf{YZ}) + E(\mathbf{XZ}) \\ &\leq 1 + 2 \min \{E(\mathbf{XY}), E(\mathbf{YZ}), E(\mathbf{XZ})\}. \end{aligned} \tag{1}$$

The above example violates inequality (1), and the non-deterministic correlations  $E(\mathbf{XY}) = E(\mathbf{YZ}) = E(\mathbf{XZ}) = -1/3$  are the lowest possible values that allow for the existence of a joint probability distribution.

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<sup>1</sup> We remark that here a quantum-like approach is impossible without additional contextual assumptions, as Alice and Bob's variables (buy stocks or not) could be used for superluminal communications [23].

### 3. QUANTUM APPROACH

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Now, let  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  correspond to certain events, say outcomes a company's stocks. For example,  $\mathbf{X} = 1$  corresponds to an increase of the stock value of company  $X$  in the following day, while  $\mathbf{X} = -1$  a decrease, and so on. Three experts, Alice ( $A$ ), Bob ( $B$ ), and Carlos ( $C$ ), have the following beliefs about those stocks. Alice is an expert on companies  $X$  and  $Y$ , but knows little or nothing about  $Z$ . Let us take the case where

$$E_A(\mathbf{XY}) = 0, \quad (2)$$

$$E_B(\mathbf{XZ}) = -\frac{1}{2}, \quad (3)$$

$$E_C(\mathbf{YZ}) = -1, \quad (4)$$

where the subscripts refer to the expectations for each of the experts. For such case, the sum of the correlations is  $-1\frac{1}{2}$ , and no joint probability distribution exists. This case is more interesting to consider that when all correlations are the same, as no obvious symmetries exist between Alice, Bob, and Carlos. Since there is no joint distribution, how can a rational decision-making agent decide what to do when faced with the question of how to bet in the market? In particular, how can she get information about the joint probability (and, in particular, about the triple moment  $E(\mathbf{XYZ})$ ) without having the joint? We will show below three possible approaches: quantum, Bayesian, and signed probabilities.

### 3 Quantum Approach

We start with a comment about the quantum-like nature of correlations (2)-(4). The random variables  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  with correlations (2)-(4) cannot be represented, in a straightforward way, by the quantum mechanical mathematical formalism. This claim can be expressed in the form of a simple proposition.

**Proposition 1.** *Let  $\hat{X}$ ,  $\hat{Y}$ , and  $\hat{Z}$  be three observables in a Hilbert space  $\mathcal{H}$  with eigenvalues  $\pm 1$ , and let them pairwise commute, and let the  $\pm 1$ -valued random variable  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  represent the outcomes of possible experiments performed on a quantum system  $|\psi\rangle \in \mathcal{H}$ . Then, there exists a joint probability distribution consistent with all the possible outcomes of  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$ .*

*Proof.* Because  $\hat{X}$ ,  $\hat{Y}$ , and  $\hat{Z}$  are observables and they pairwise commute, it follows that their combinations,  $\hat{X}\hat{Y}$ ,  $\hat{Y}\hat{Z}$ ,  $\hat{X}\hat{Z}$ , and  $\hat{X}\hat{Y}\hat{Z}$  are also observables, and they commute with each other. For instance,

$$\left(\hat{X}\hat{Y}\hat{Z}\right)^\dagger = \hat{Z}^\dagger\hat{Y}^\dagger\hat{X}^\dagger = \hat{X}\hat{Y}\hat{Z}.$$

Furthermore,

$$[\hat{X}\hat{Y}\hat{Z}, \hat{X}] = [\hat{X}\hat{Y}\hat{Z}, \hat{Y}] = \dots = [\hat{X}\hat{Y}\hat{Z}, \hat{X}\hat{Z}] = 0.$$

Therefore, quantum mechanics implies that all three observables  $\hat{X}$ ,  $\hat{Y}$ , and  $\hat{Z}$  can be simultaneously measured. Since this is true, for the same state  $|\psi\rangle$  we

can create a full data table with all three values of  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  (i.e., no missing values), which implies the existence of a joint.<sup>2</sup>

So, how would a quantum-like model of the triple be like? The above result depends on the use of the same quantum state  $|\psi\rangle$  throughout the many runs of the experiment, and to circumvent it we would need to use different states for the system. In other words, if we want to use a quantum formalism to describe the correlations (2)-(4),  $|\psi\rangle$  would have to be selected for each run, such that a different state would be used when we measure  $\hat{X}\hat{Y}$ , e.g.  $|\psi\rangle_{xy}$ , than when we measure  $\hat{X}\hat{Z}$ , e.g.  $|\psi\rangle_{xz}$ . Such changes in state could include additional correlations between the variables. However, this quantum mechanical approach is not only *ad hoc*, but does not address the question about the triple moment, as it is not clear how to get it from the formalism.

In fact, the quantum approach above could be similarly implemented using a contextual theory. For instance, Dzhafarov [25,26,27] proposes the use of an extended probability space where different random variables (say,  $\mathbf{X}_z$  and  $\mathbf{X}_y$ ) are used, and where we then ask how similar they are to each other (for instance, what is the value of  $P(\mathbf{X}_z \neq \mathbf{X}_y)$ ). However, as with the quantum case, the meaning given to  $P(\mathbf{X} = 1)$  in our example does not fit with this model, as it corresponds to the expectation of an increase in the stock value of company  $X$  in the future, and the  $X$  that Alice is talking about is exactly the same one for Bob and Carlos, as it corresponds to the increase in the objective value (in the future) of a stock in the same company. Furthermore, as expected due to its similar features, this approach has the same problem as the quantum one in terms of dealing with the triple moment, but it has the advantage of making it clearer what the problem is: the triple moment does not exist because we have nine random variables instead of three, as we have three different contexts.

## 4 Bayesian Approach

In the example from Section 2 all correlations and expectations are given, but we do not have the triple moment  $E(\mathbf{XYZ})$ . Furthermore, since we do not have a joint probability distribution, we cannot compute the range of values for such moment based on the expert's beliefs. But the question still remains as to what would be our best bet given what we know, i.e., what is our best guess for  $E(\mathbf{XYZ})$ . There are many different ways to approach this problem, such as paraconsistent logics, consensus reaching, or information revision to restore consistency. Common to all those approaches is the complexity of how to resolve the inconsistencies, often with the aid of *ad hoc* assumptions [22]. Here we briefly sketch how one could have a Bayesian approach for this issue [28,29].

<sup>2</sup> An attentive reader might be puzzled by our result above, as it seems to contradict Bell's use of three settings of detection apparatuses to prove his inequalities [5]. Though Bell used three settings, four observables were necessary, and they do not all pairwise commute (for example, for the standard  $A$ ,  $A'$ ,  $B$ , and  $B'$ ,  $[A, A'] \neq 0$ ).

#### 4. BAYESIAN APPROACH

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In the Bayesian approach, the decision maker, Deanna ( $D$ ), needs to access what is the joint probability distribution from a set of expectations that are conflicting. To set the notation, let us first look at the case when there is only one expert. Let  $P_A(x) = P_A(\mathbf{X} = x|\delta_A)$  be the probability assigned to event  $x$  by Alice conditioned on Alice's knowledge  $\delta_A$ , and let  $P_D(x) = P_D(\mathbf{X} = x|\delta_D)$  be Deanna's prior distribution, also conditioned on her knowledge  $\delta_D$ . Furthermore, let  $\mathbf{P}_A = P_A(x)$  be a continuous random variable,  $\mathbf{P}_A \in [0, 1]$ , such that its outcome is  $P_A(x)$ . The idea behind  $\mathbf{P}_A$  is that consulting an expert is similar to conducting an experiment where we sample the experts opinion by observing a distribution function, and therefore we can talk about the probability that an expert will give an answer for a specific sample point. Then, for this case, Bayes's theorem can be written as

$$P'_D(x|\mathbf{P}_A = P_A(x)) = \frac{P_D(\mathbf{P}_A = P_A(x)) P_D(x)}{P_D(\mathbf{P}_A = P_A(x))},$$

where  $P'_D(x|\mathbf{P}_A = P_A(x))$  is Deanna's posterior distribution revised to take into account the expert's opinion. As is the case with Bayes's theorem, the difficulty lies on determining the likelihood function  $P_D(\mathbf{P}_A)$ . To do so, Deanna first needs to assume what is the distributions for Alice. Notice that when we talk about the likelihood function  $P_D(\mathbf{P}_A)$ , we are talking about Deanna's probabilities for a given function  $\mathbf{P}_A$ . For the simple case where  $x$  are the values of a  $\pm 1$ -valued random variable  $\mathbf{X}$ , an example could be a  $\mathbf{P}_A$  that can take the values  $P_A(\mathbf{X} = 1) = 1/2$  or  $P_A(\mathbf{X} = 1) = 1/4$ , and Deanna then assigns probabilities to each one of those distributions. So, this likelihood function is, in a certain sense, Deanna's model of Alice, as it is what Deanna believes are the likelihoods of each of Alice's beliefs. For example, she can assume that Alice herself is Bayesian, and a Bernoulli distribution would be adequate. In other words, she should have a model of the experts. Such model of experts is akin to giving each expert a certain measure of credibility, since an expert whose model doesn't fit Deanna's would be assigned lower probability than an expert whose model fits.

The extension for our case of three experts and three random variables is cumbersome but straightforward. For Alice, Bob, and Carlos, Deanna needs to have a model for each one of them, based on her prior knowledge about  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$ , as well as Alice, Bob, and Carlos. Following Morris [28], we construct a set  $E$  consisting of our three experts joint priors:

$$E = \{P_A(x, y), P_B(y, z), P_C(x, z)\}.$$

Deanna's is now faced with the problem of determining the posterior  $P'_D(x|E)$ , using Bayes's theorem, given her new knowledge of the expert's priors.

In a Bayesian approach, the decision maker should start with a prior belief on the stocks of  $X$ ,  $Y$ , and  $Z$ , based on her knowledge. Suppose Deanna has no knowledge about  $X$ ,  $Y$ , and  $Z$ , and therefore assigns no correlations between their stocks as her prior distribution. Let us use the following notation for the probabilities of each atom:

$$p_{xyz} = P(\mathbf{X} = +1, \mathbf{Y} = +1, \mathbf{Z} = +1),$$

$$p_{xy\bar{z}} = P(\mathbf{X} = +1, \mathbf{Y} = +1, \mathbf{Z} = -1), p_{\bar{x}y\bar{z}} = P(\mathbf{X} = -1, \mathbf{Y} = +1, \mathbf{Z} = -1), \text{ and}$$

so on. Then Deanna's prior probabilities for the atoms are

$$p_{xyz}^D = p_{\bar{x}yz}^D = \dots = p_{x\bar{y}\bar{z}}^D = \frac{1}{16},$$

where the superscript  $D$  refers to Deanna. Furthermore, when reasoning about the likelihood function, Deanna asks what would be the probable distribution of responses of Alice if somehow she (Deanna) could see the future and find out that  $E(XY) = -1$ . For such case, it would be reasonable for Alice to find it more probable to have, say,  $\bar{x}y$  than  $xy$ . So, in terms of the correlation  $\epsilon_A$ , Deanna could assign the following likelihood function:

$$P_D(\epsilon_A|\bar{x}y) = P_D(\epsilon_A|x\bar{y}) = \frac{1}{4}(1 - \epsilon_A)^2, \quad (5)$$

$$P_D(\epsilon_A|xy) = P_D(\epsilon_A|\bar{x}\bar{y}) = 1 - \frac{1}{4}(1 - \epsilon_A)^2, \quad (6)$$

where the minus sign represents the negative, i.e.  $p_{xy\cdot}^A = p_{x\bar{y}\cdot} = \frac{1}{4}(1 + \epsilon_A)$  and  $p_{\bar{x}y\cdot} = p_{x\bar{y}\cdot} = \frac{1}{4}(1 - \epsilon_A)$ . So, Deanna's posterior, once she knows that Alice thought the correlation to be zero (cf. (2)), constitutes, as we mentioned above, an experiment. To illustrate the computation, we compute below the value of  $p'_{xyz}$ . From Bayes's theorem

$$\begin{aligned} p_{xyz}^{D|A} &= k \frac{\left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8}}{\left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[\frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8}} \\ &= k \left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] = \frac{3}{16}, \end{aligned}$$

where the normalization constant  $k$  is given by

$$\begin{aligned} k^{-1} &= \left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[\frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[\frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} \\ &\quad + \left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[\frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[\frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} \\ &\quad + \left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8} + \left[1 - \frac{1}{4}(1 - \epsilon_A)^2\right] \frac{1}{8}, \end{aligned}$$

and we use the notation  $p^{D|A}$  to explicitly indicate that this is Deanna's posterior probability informed by Alice's expectation. Similarly, we have

$$p_{\bar{x}yz}^{D|A} = p_{\bar{x}y\bar{z}}^{D|A} = p_{x\bar{y}z}^{D|A} = p_{x\bar{y}\bar{z}}^{D|A} = \frac{1}{16},$$

and

$$p_{xyz}^{D|A} = p_{xy\bar{z}}^{D|A} = p_{x\bar{y}z}^{D|A} = p_{x\bar{y}\bar{z}}^{D|A} = \frac{3}{16}.$$

## 5. SIGNED PROBABILITIES

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If we apply Bayes's theorem twice more, to take into account Bob's and Carlos's expert opinions from (3) and (4), using likelihood functions similar to the one above, we compute the following posterior

$$p_{xyz}^{D|ABC} = p_{x\bar{y}z}^{D|ABC} = p_{x\bar{y}\bar{z}}^{D|ABC} = p_{xy\bar{z}}^{D|ABC} = 0,$$

$$p_{\bar{x}yz}^{D|ABC} = p_{\bar{x}\bar{y}z}^{D|ABC} = \frac{7}{68},$$

and

$$p_{xy\bar{z}}^{D|ABC} = p_{\bar{x}y\bar{z}}^{D|ABC} = \frac{27}{68}.$$

Finally, from the joint, we can compute all the moments, including the triple moment  $E(\mathbf{XYZ}) = 0$ . In fact, if we compute Deanna's posterior distribution for any values of the correlations  $\epsilon_A$ ,  $\epsilon_B$ , and  $\epsilon_C$ , we obtain the same triple moment, as it was encoded in Deanna's prior distribution.

We emphasize at this point that the value for the triple moment comes out of not only of Deanna's prior distribution, but also from her model of what an expert behavior is. For instance, the zero probabilities of  $p_{xyz}^{D|ABC}, \dots, p_{\bar{x}\bar{y}\bar{z}}^{D|ABC}$  are a consequence of (5) and (6). So, the Bayesian approach, though providing a proper distribution for the atoms, does not in any way reflect the uncertainty of the inference.

## 5 Signed Probabilities

We now want to see how we can use signed probabilities to approach the same problem as before. It seems that the first person to seriously consider using signed probabilities was Dirac in his Bakerian Lectures on the physical interpretation of relativistic quantum mechanics [30]. Ever since, many physicists, most notably Feynman [31], tried to use them, with limited success, to describe physical processes (see [32] or [33] and references therein). The main problem with signed (or negative) probabilities is its lack of a clear interpretation, which limits its use as a purely computational tool. But, concluded Feynman, even as a computation tool, signed probabilities seem to have no use. It is the goal of this section to show that, at least for some social phenomena, signed probabilities can be useful.

Assuming the existence of a joint probability distribution, we can determine the probability for each atom. Then, we have the following equations for the atoms.

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} + p_{xy\bar{z}} + p_{x\bar{y}\bar{z}} + p_{\bar{x}y\bar{z}} + p_{\bar{x}\bar{y}z} + p_{\bar{x}\bar{y}\bar{z}} = 1, \quad (7)$$

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\bar{y}\bar{z}} - p_{\bar{x}y\bar{z}} - p_{\bar{x}\bar{y}z} - p_{\bar{x}\bar{y}\bar{z}} = 0, \quad (8)$$

$$p_{xyz} + p_{\bar{x}yz} - p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\bar{y}\bar{z}} + p_{\bar{x}y\bar{z}} - p_{\bar{x}\bar{y}z} - p_{\bar{x}\bar{y}\bar{z}} = 0, \quad (9)$$

$$p_{xyz} + p_{\bar{x}yz} + p_{x\bar{y}z} - p_{xy\bar{z}} - p_{x\bar{y}\bar{z}} - p_{\bar{x}y\bar{z}} + p_{\bar{x}\bar{y}z} - p_{\bar{x}\bar{y}\bar{z}} = 0, \quad (10)$$



$$p_{xyz} - p_{\bar{x}yz} - p_{x\bar{y}z} + p_{xy\bar{z}} - p_{x\bar{y}\bar{z}} - p_{\bar{x}y\bar{z}} + p_{\bar{x}\bar{y}z} + p_{\bar{x}\bar{y}\bar{z}} = 0, \quad (11)$$

$$p_{xyz} - p_{\bar{x}yz} + p_{x\bar{y}z} - p_{xy\bar{z}} - p_{x\bar{y}\bar{z}} + p_{\bar{x}y\bar{z}} - p_{\bar{x}\bar{y}z} + p_{\bar{x}\bar{y}\bar{z}} = -\frac{1}{2}, \quad (12)$$

$$p_{xyz} + p_{\bar{x}yz} - p_{x\bar{y}z} - p_{xy\bar{z}} + p_{x\bar{y}\bar{z}} - p_{\bar{x}y\bar{z}} - p_{\bar{x}\bar{y}z} + p_{\bar{x}\bar{y}\bar{z}} = -1, \quad (13)$$

where equation (7) comes from the fact that all probabilities must sum to one, equations (8)-(10) from the zero expectations, and equations (11)-(13) from the pairwise correlations. Of course, this problem is underdetermined, as we have seven equations and eight unknowns. A general solution to it is

$$\begin{aligned} p_{xyz} &= -p_{\bar{x}yz} = -\frac{1}{8} - \delta, \\ p_{x\bar{y}z} &= p_{\bar{x}y\bar{z}} = \frac{3}{16}, \\ p_{xy\bar{z}} &= p_{\bar{x}\bar{y}z} = \frac{5}{16}, \\ p_{x\bar{y}\bar{z}} &= -p_{\bar{x}y\bar{z}} = -\delta, \end{aligned}$$

where  $\delta$  is a real number. It is immediate from the above equations that for any value of  $\delta$  the probabilities are negative, as expected. First, we notice that we can use the joint probability distribution to compute the expectation of the triple moment, which is

$$\begin{aligned} E(\mathbf{XYZ}) &= p_{xyz} - p_{\bar{x}yz} - p_{x\bar{y}z} - p_{xy\bar{z}} + p_{x\bar{y}\bar{z}} + p_{\bar{x}y\bar{z}} + p_{\bar{x}\bar{y}z} - p_{\bar{x}\bar{y}\bar{z}} \\ &= -\frac{1}{8} - \delta - \frac{1}{8} - \delta - \frac{3}{16} - \frac{5}{16} - \delta + \frac{3}{16} + \frac{5}{16} - \delta \\ &= -\frac{1}{4} - 4\delta. \end{aligned}$$

Since  $-1 \leq E(\mathbf{XYZ}) \leq -1$ , it follows that  $-1\frac{1}{4} \leq \delta \leq \frac{3}{4}$ . Of course,  $\delta$  is not determined by the lower moments, as we should expect, but we can impose further constraints. For instance, let us now define the total negative mass as

$$M^- = -\frac{1}{2} \sum_i (p_i - |p_i|),$$

where  $p_i, i \in \{xyx, \bar{x}yz, \dots, \bar{x}\bar{y}\bar{z}\}$ , is the probability for the atoms. Intuitively,  $M^-$  is a measure of how a given negative joint probability distribution departs from a Kolmogorovian distribution. Therefore, if we want our distribution to be as close as a classical one, we should minimize the negative mass  $M^-$ . In the context of our problem, we could think of minimizing  $M^-$  as trying to get as close as possible to a hypothetical consistent measure of beliefs based on the conclusions that experts would reach if they were able to analyze all possible information (including the opinion of other experts).

So, to minimize  $M^-$ , we focus only on the terms that contribute to it: the negative ones. To do so, let us split the problem into several different sections. Let us start with  $\delta \geq 0$ , which gives

$$M_{\delta \geq 0}^- = -\frac{1}{8} - 2\delta,$$

## 6. CONCLUSIONS

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having a minimum of  $-\frac{1}{8}$  when  $\delta = 0$ . For  $-1/8 \leq \delta < 0$ ,

$$M_{-\frac{1}{8} \leq \delta < 0}^- = \delta - \frac{1}{8} + \delta = -\frac{1}{8},$$

which is a constant value. Finally, for  $\delta < -1/8$ , the mass for the negative terms is given by

$$M_{\delta < -\frac{1}{8}}^- = \frac{1}{8} - 2\delta.$$

Therefore, negative mass is minimized when  $\delta$  is in the following range

$$-\frac{1}{8} \leq \delta \leq 0.$$

Now, going back to the triple correlation, we see that by imposing a minimization of the negative mass we restrict its values to the following range:

$$-\frac{1}{4} \leq E(\mathbf{XYZ}) \leq \frac{1}{2}.$$

But equations (7)-(13) and the fact that the random variables are  $\pm 1$ -valued allow any correlation between  $-1$  and  $1$ , and we see that the minimization of the negative mass offers further constraints to a decision maker. Notice that the interval given is consistent with the value predicted by the Bayesian model.

## 6 Conclusions

The use of the quantum mechanical formalism has been successfully done in many distinct examples in the social sciences. However, one of the questions we raised was whether some minimalist versions of the quantum formalism which do not include a full version of Hilbert spaces and observables could be relevant. In this paper we extended the simple example modeled with neural oscillators in [20] to a different case where each random variable could be interpreted as outcomes of a market, and where the inconsistencies between the correlations could be interpreted as inconsistencies between experts' beliefs. Such inconsistencies result in the impossibility to define a standard probability measure that allows the decision-maker to select an expectation for the triple moment. Using signed probabilities inspired by quantum mechanics, we showed that a (non-Kolmogorovian) joint probability distribution could be computed. We then defined the negative mass of such distribution, and interpreted it as a departure from a classical framework. By minimizing the negative mass, we showed that the triple moment had extra constraints that did not come from the marginal distributions. We then proposed that signed probabilities could be used as a normative model for decision making based on inconsistent beliefs.

As we briefly discussed in Section 4, a Bayesian approach provides a way to make decisions based on the same set of inconsistent beliefs. However, the Bayesian approach requires not only a prior distribution, but also a model of the

expert's opinions by the decision maker. As such, the Bayesian model constructed here gives a triple moment that is consistent with the signed probability bounds. However, we do emphasize that the signed probability approach can be computed much more directly, without any additional assumptions, and could provide a fast way to estimate possible bounds on rational decisions.

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## References

1. Khrennikov, A.: Ubiquitous Quantum Structure. Springer Verlag, Heidelberg (2010)
2. Haven, E., Khrennikov, A.: Quantum Social Science. Cambridge University Press, Cambridge, UK (2013)
3. Birkhoff, G., Neumann: The logic of quantum mechanics. *Annals of Mathematics* **37**(4) (October 1936) 823–843
4. Weizsäcker: Probability and quantum mechanics. *The British Journal for the Philosophy of Science* **24**(4) (December 1973) 321–337
5. Bell, J.S.: On the {e}instein-odolsky-osen paradox. *Physics* **1**(3) (1964) 195–200
6. Bell, J.S.: On the problem of hidden variables in quantum mechanics. *Rev. Mod. Phys.* **38**(3) (1966) 447–452
7. Kochen, S., Specker, E.P.: The problem of hidden variables in quantum mechanics. In Hooker, C.A., ed.: *The Logico-Algebraic Approach to Quantum Mechanics*. D. Reidel Publishing Co., Dordrecht, Holland (1975) 293–328
8. Greenberger, D.M., Horne, M.A., Zeilinger, A.: Bell's theorem, quantum theory, and conceptions of the universe. M. Kafatos ed (1989) 69–72
9. de Barros, J.A., Suppes, P.: Inequalities for dealing with detector inefficiencies in {g}reenberger-orne-zeilinger type experiments. *Physical Review Letters* **84** (2000) 793–797
10. Pitowsky, I.: Quantum probability, quantum logic. Number 321 in *Lecture Notes in Physics*. Springer Verlag, Berlin (1989)
11. Holik, F., Plastino, A., Sáenz, M.: A discussion on the origin of quantum probabilities. arXiv:1211.4952 (November 2012)
12. de Barros, J.A., Suppes, P.: Probabilistic results for six detectors in a three-particle GHZ experiment. In Bricmont, J., Dürr, D., Galavotti, M.C., Ghirardi, G., Petruccione, F., Zanghi, N., eds.: *Chance in Physics*. Volume 574 of *Lectures Notes in Physics*. (2001) 213
13. de Barros, J.A., Suppes, P.: Probabilistic inequalities and upper probabilities in quantum mechanical entanglement. *Manuscrito* **33** (2010) 55–71

## 6. CONCLUSIONS

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14. Suppes, P., Zanotti, M.: When are probabilistic explanations possible? *Synthese* **48**(2) (1981) 191–199
15. Suppes, P., de Barros, J.A., Oas, G.: A collection of probabilistic hidden-variable theorems and counterexamples. In Pratesi, R., Ronchi, L., eds.: *Waves, Information, and Foundations of Physics: a tribute to Giuliano Toraldo di Francia on his 80th birthday*, Florence, Italy, Italian Physical Society (October 1996)
16. de Barros, J.A., Suppes, P.: Quantum mechanics, interference, and the brain. *Journal of Mathematical Psychology* **53**(5) (2009) 306–313
17. Dieks, D.: Communication by EPR devices. *Physics Letters A* **92**(6) (November 1982) 271–272
18. Suppes, P., de Barros, J.A., Oas, G.: Phase-oscillator computations as neural models of stimulus–response conditioning and response selection. *Journal of Mathematical Psychology* **56**(2) (April 2012) 95–117
19. de Barros, J.A.: Quantum-like model of behavioral response computation using neural oscillators. *Biosystems* **110**(3) (December 2012) 171–182
20. de Barros, J.A.: Joint probabilities and quantum cognition. *AIP Conference Proceedings* **1508**(1) (December 2012) 98–107
21. da Costa, N.C.A.: On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic* **15**(4) (October 1974) 497–510
22. Genest, C., Zidek, J.V.: Combining probability distributions: A critique and an annotated bibliography. *Statistical Science* **1**(1) (February 1986) 114–135 *Mathematical Reviews number (MathSciNet): MR833278*.
23. Oas, G., de Barros, J.A., Carvalhaes, C.G., Suppes, P.: Exploring non-signaling polytopes with joint quasi- probability distributions, Vaxjo, Sweden (June 2013)
24. Suppes, P., Zanotti, M.: A new proof of the impossibility of hidden variables using the principles of exchangeability and identity of conditional distributions. *Studies in the Foundations of Quantum Mechanics* (1980) 173–191
25. Dzhafarov, E.N.: Selective influence through conditional independence. *Psychometrika* **68**(1) (March 2003) 7–25
26. Dzhafarov, E.N., Kujala, J.V.: Selectivity in probabilistic causality: Where psychology runs into quantum physics. *Journal of Mathematical Psychology* **56**(1) (February 2012) 54–63
27. Dzhafarov, E.N., Kujala, J.V.: All-possible-couplings approach to measuring probabilistic context. *PLoS ONE* **8**(5) (May 2013) e61712
28. Morris, P.A.: Decision analysis expert use. *Management Science* **20**(9) (May 1974) 1233–1241
29. Morris, P.A.: Combining expert judgments: A bayesian approach. *Management Science* **23**(7) (March 1977) 679–693
30. Dirac, P.A.M.: Bakerian lecture. the physical interpretation of quantum mechanics. *Proceedings of the Royal Society of London B* **A180** (1942) 1–40
31. Feynman, R.P.: Negative probability. In Hiley, B.J., Peat, F.D., eds.: *Quantum implications: essays in honour of David Bohm*. Routledge, London and New York (1987) 235–248
32. Mückenheim, G.: A review of extended probabilities. *Physics Reports* **133**(6) (1986) 337–401
33. Khrennikov, A.: *Interpretations of Probability*. Walter de Gruyter (February 2009)