Comments on the Quantum Potential Approach to a Class of Quantum Cosmological Models

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Abstract

In this comment we bring attention to the fact that when we apply the ontological interpretation of quantum mechanics, we must be sure to use it in the coordinate representation. This is particularly important when canonical tranformations that mix momenta and coordinates are present. This implies that some of the results obtained by A. Błaut and J. Kowalski-Glikman are incorrect.

1 Introduction

In a recent paper, A. Błaut and J. Kowalski-Glikman [1] tried to interpret a classe of quantum cosmological models in terms of Bohm's causal interpretation of quantum mechanics [2]. Following a formalism developed by Ashtekar *et al.* [3], they applied a canonical quantization procedure to a restricted class of spacetimes, whose Hamiltonian constraint has been put in a simple form after a non-trivial canonical transformation. Then, with the standard decomposition of the wavefunction in polar form, they obtained from the Wheeler-DeWitt equation a modified Hamilton-Jacobi equation with an extra quantum potential term. From the solutions of this equation, they computed Bohmian trajectories, and obtained possible scenarios for the universe modeled by the given wavefunction. In this comment, we will show that the interpretation presented in reference [1] is not adequate because when using Bohm's interpretation we have to make sure that the wavefunction we use is in the coordinate representation.

2 The Classical Model

The minisuperspace examples A. Błaut and J. Kowalski-Glikman [1] used were classes of diagonal, spatially homogeneous cosmological models which admit intrinsic, multiply transitive symmetry groups (DIMT models). The spatially homogeneous diagonal 4-metric can be expressed in the form

$$ds^2 = -N^2(t)\,dt^2 + \sum_{i=1}^{3} \exp(2\beta^i)(\omega^i)^2 \tag{1}$$

where $N(t)$ is the lapse function, and ω^i is a basis of spatial 1-forms which are left invariant by the action of the isometry group.

Since in their paper, A. Błaut and J. Kowalski-Glikman analyzed the case of a plane wave in a Bianchi type IX spacetime, we will focus our attention on this case. Imposing the Taub gauge $N_t = 12\exp(3\beta^0)$, we can express the scalar Hamiltonian constraint for the Bianchi IX model as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_+, \tag{2}$$

where

$$\mathcal{H}_0 = -\frac{1}{2}\bar{p}_0^2 - 24\exp(2\sqrt{3}\bar{\beta}^0), \tag{3}$$

$$\mathcal{H}_+ = \frac{1}{2}\bar{p}_+^2 + 6\exp(-4\sqrt{3}\bar{\beta}^+). \tag{4}$$

In the above expressions we have

$$\left(\beta^1, \beta^2, \beta^3\right) = \sqrt{3}\left(\bar{\beta}^0 + \bar{\beta}^+, \bar{\beta}^0 + \bar{\beta}^+, -\bar{\beta}^+\right). \tag{5}$$

The separable form of the scalar constraint presented above makes it possible to perform a canonical transformation that simplify its form. It is given by

$$\tilde{p}_A = \sqrt{\bar{p}_A^2 + a\exp(2b\bar{\beta}^A)}, \tag{6}$$

and

$$\tilde{\beta} = \frac{1}{b}\left[\log\left(-\bar{p} + \sqrt{p^2 + a\exp(2b\bar{\beta})}\right) - \log\left(\sqrt{a}\exp(b\bar{\beta})\right)\right]. \tag{7}$$

where a and b can be read from equtions (3) and (4). With the transformations (6) and (7), the Hamiltonian constraint (2) for the Bianchi IX model acquires the simple form

$$\mathcal{H} = -\frac{1}{2}\left(\tilde{p}_0^2 - \tilde{p}_+^2\right) = 0. \tag{8}$$

3 Bohm's Trajectories

Let us start with the equation

$$\hat{\mathcal{H}}\psi(\bar{\beta}^A) = \left[\frac{1}{2}\Box + \mathcal{V}(\bar{\beta}^A)\right]\psi(\bar{\beta}^A) = 0 \tag{9}$$

where $\Box \equiv \eta^{AB}\partial/\partial\bar{\beta}^A\partial/\partial\bar{\beta}^B$ with $\eta^{AB} = \mathrm{diag}(1,-1)$ and

$$\mathcal{V}(\bar{\beta}^A) = -24\exp(2\sqrt{3}\bar{\beta}^0) + 6\exp(-4\sqrt{3}\bar{\beta}^+). \tag{10}$$

Equation (9) is the the Wheeler-de Witt equation of the Bianchi IX model coming from the Hamiltonian constraint (2). We will use the standard polar decomposition for the wavefunction

$$\psi = R(\bar{\beta}^A)\exp(\frac{i}{\hbar}S(\bar{\beta}^A)). \tag{11}$$

Substitution of ψ in the form (11) into (9) results in

$$-\frac{1}{2}\eta^{AB}\frac{\partial S}{\partial\bar{\beta}^A}\frac{\partial S}{\partial\bar{\beta}^B} + \mathcal{V} - \frac{\hbar^2}{2}\frac{1}{R}\Box R = 0,$$

which can be seen as a Hamilton-Jacobi like equation plus a quantum potential term. Adopting the Bohm's interpretation, one can *postulate* the momenta as

$$\bar{p}_A = \frac{\partial S}{\partial\bar{\beta}^A} = \eta_{AB}\frac{\mathrm{d}\bar{\beta}^B}{\mathrm{d}t}, \tag{12}$$

where the parameter t, the time, was introduced. The trajectories followed by the system are solutions of the equation (12), and are different from the classical ones due to the quantum potential.

In the tilde variables, the Wheeler-DeWitt equation becomes (see equation (8))

$$\frac{1}{2}\Box'\psi = 0, \tag{13}$$

where $\frac{1}{2}\Box' \equiv \eta^{AB}\partial/\partial\tilde{\beta}^A\partial/\tilde{\beta}^B$, which is evidently much more simple to solve then equation (9).

4 The Plane Wave Example

In this Section we will analyze the example given by Błaut and Kowalski-Glikman. The wavefunction they interpreted is given by

$$\psi(\tilde{\beta}_0, \tilde{\beta}_+) = \exp[i(k+l)U] + \exp[i(k-l)V], \tag{14}$$

where $U = \tilde{\beta}_0 + \tilde{\beta}_+$, $V = \tilde{\beta}_0 - \tilde{\beta}_+$, and k and l are real constants. It is a solution of the Wheeler-De Witt equation (13) in the tilde variables. They obtained Bohmian trajectories by using equation (12) in the tilde variables

$$\tilde{p}_A = \frac{\partial S}{\partial \tilde{\beta}^A} = \eta_{AB} \frac{\mathrm{d}\tilde{\beta}^B}{\mathrm{d}t}, \tag{15}$$

where S is the phase of the wave function (14). Our main point is that the tilde variables were obtained from the barred variables by a non-trivial canonical transformation which mix momenta with coordinates, and hence one cannot apply directly the Bohm interpretation to these variables by using equation (15).

Quantum mechanically, to look for a canonical transformation means to look for a unitary transformation that maps wavefunctions from the original set of variables to the new one. In other words, we need to find the kernel $\langle\tilde{\beta}|\bar{\beta}\rangle$, which was obtained in reference [1]. However, the canonical transformation (6) and (7) to the tilde variables mix coordinates with momenta. As is well known, Bohm's intepretation only makes sense in coordinate representation [4, 5]. If we make a canonical tranformation that mixes momenta and coordinates, we may end up having misleading results. The only safe way to guarantee the correct interpretation of the solution (14) is to go back to the wave function expressed in the original set of variables by using the kernel $\langle\tilde{\beta}|\bar{\beta}\rangle$, and then use equation (12) to obtain the quantum trajectories. The final result will in general be different from the one obtained directly from the wave function (14) by using equation (15) in the tilde variables and then going back to the barred variables by using the inverse of the tranformation (6) and (7), as is done in reference [1].

Let us illustrate this point with a simple example showing how the two procedures can give different Bohmian trajectories. Take the Hamiltonian for the free particle

$$H = \frac{p^2}{2m}. \tag{16}$$

and let us make a canonical transformation to a new set of variables given by

$$X = \frac{a^2}{\hbar} p + x, \tag{17}$$

$$P = p. \tag{18}$$

where a is some constant with dimension of length. Then, in the new set of variables we have

$$H = \frac{P^2}{2m}.$$

We want to find a kernel that transforms the wavefunction from the original set of variables to the new one. This is accomplished by solving the following set of equations:

$$\hat{p}\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}X \, \langle x|X\rangle \hat{P}\psi(X) \tag{19}$$

$$\hat{x}\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}X \, \langle x|X\rangle (\hat{X} - \hat{P})\psi(X). \tag{20}$$

We can easily solve these equations and obtain that

$$\langle x|X\rangle = e^{-\frac{i}{2a^2}(x-X)^2}. \tag{21}$$

Now we can look for a particular solution of the free particle Schroedinger equation. In the X, P coordinates, one possible solution is the gaussian

$$\psi(X,t) = b(t) \exp\left[-X^2 \left(\frac{a^2 - ic(t)}{a^4 + c(t)^2}\right)\right],$$

where $b(t) = \{2/[\pi a^2(1 + ic(t)/a^2)^2]\}^{1/4}$, and $c(t) = 2\hbar t/m$. If we set, to simplify the computations, $2\hbar = m = a^2 = 1$, we obtain from

$$P = \dot{X} = \frac{\partial S}{\partial X}$$

the result

$$X(t) = \beta(1 + t^2)^{1/2},$$

where β is an integration constant. Using now equations (17) and (18) we obtain

$$x(t) = X(t) - 2P(t) = \frac{\beta(t-1)^2}{(t^2+1)^{1/2}}. \tag{22}$$

On the other hand, if we make the transformation

$$\psi(x,t) = b(t)\int \exp\left[-\frac{i}{2a^2}(x-X)^2\right]\exp\left[-X^2\left(\frac{a^2-ic(t)}{a^4+c(t)^2}\right)\right]\mathrm{d}X$$

we get at once

$$\psi(x,t) = b'(t)\exp\left\{-x^2\left[\frac{a^2-i(c^2(t)-2a^2)}{(c^2(t)-2a^2)^2+a^4}\right]\right\}, \tag{23}$$

where $b'(t)$ is a function of time which is not important for what follows. Setting again $2\hbar = m = a^2 = 1$, we get from the equation

$$p = \dot{x} = \frac{\partial S}{\partial x}$$

the solution

$$x(t) = \beta'[(t-2)^2+1]^{1/2}, \tag{24}$$

where β' is a constant of integration, which is different from the solution (22)[1]. Hence, the two methods are inequivalent.

In conclusion, we must be very careful when we use the causal interpretation because the Bohmian trajectories are not invariant under general canonical transformations. Knowing this fact, what we have to do in the Bianchi IX example is to map the wave function in the tilde variables into the correspondent solution in the barred variables, which are the configuration variables related to the physical metric, and only after use the causal interpretation to find the Bohmian trajectories. Applying directly the causal interpretation to the wave function in the tilde variables yields wrong Bohmian trajectories in the barred coordinates.

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[1] Note that solution (24) is consistent with the gaussian (23) while solution (22) is not (see reference [5], chapter 4).

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