Ontological Indistinguishability as a central tenet of quantum theory

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Abstract

Quantum indistinguishability directly relates to the philosophical debate on the notions of identity and individuality. They are crucial for our understanding of multipartite quantum systems. Furthermore, the correct interpretation of this feature of quantum theory has implications that transcend fundamental science and philosophy, given that quantum indistinguishability is a resource in quantum information theory. Most of the conceptual analysis of quantum indistinguishability is restricted to studying the permutational invariance of quantum states, the concomitant quantum statistics, and their entanglement. Here, we analyze the role of indistinguishability and nonindividuality in other areas of quantum theory. We start by analyzing how a very peculiar use of indistinguishability underlies Feynman's rules for summing amplitudes in interference phenomena. Next, we study how quantum indistinguishability is underestimated in several topics of debate in the quantum physics literature, such as the EPR argument, Bell's inequalities, and the Bell-KS theorem. Finally, we argue that an ontology of truly indistinguishable entities can serve as a basis for a quantum ontology that can give interesting answers to the interpretational problems of quantum mechanics. We claim that, in addition to superposition, contextuality, and entanglement, indistinguishability (understood in a robust ontological sense) is one of the central features of quantum physics.

1 Introduction

What is the most important feature of quantum physics? Some authors, such as Erwin Schrödinger, suggest entanglement [1]. Others mention the superposition principle. One could say that quantum physics is characterized by both the superposition principle and entanglement.

But there is another feature that cannot be reduced to these two principles: indistinguishability. The fact that there are situations in which there is a fundamental impossibility to decide empirically which quantum system is which has no analog with classical entities such as billiard balls [2]. We can always attribute identities and labels to such classical objects. There are, of course, circumstances in which we simply do not know which is which and have no means to decide. But it does not matter how similar they are. We can always, in principle, envisage an operational procedure to distinguish two given billiard balls. Classical objects appear to us as indistinguishable in an epistemological way. But the indistinguishability of quantum systems seems to have a deeper ontological meaning. If quantum theory is correct for quantum systems entering the indistinguishability regime, there is no operational procedure to distinguish them [3]. Thus, together with superposition and entanglement, one could add indistinguishability as one of the main features of quantum theory.

In this work, we take a step further and propose the following hypothesis. Quantum indistinguishability should be the basis of an ontology for quantum theory [4]. By accepting the hypothesis that quantum systems are truly indiscernible, one can shed light on many quantum features that otherwise would seem disconnected. For instance, indistinguishability is usually not considered in philosophical discussions of the Kochen-Specker contradiction [5] or the EPR paradox [6]. In particular, in most arguments giving place to quantum paradoxes, it is tacitly assumed that the objects involved obey the classical laws of identity. We will argue that this is problematic, since the assumption of identity for quantum systems is very strong and should not be taken lightly.

We will show that many of the so-called quantum paradoxes do not follow if quantum objects are considered truly indiscernible and do not obey the classical laws of identity. On the positive side, it is possible to use indistinguishability as a unifying ontological principle for quantum theory. In particular, quantum indistinguishability seems to be deeply related to quantum contextuality [7, 4].

After revisiting the standard formulation of quantum indistinguishability in Section 2, we analyze the connection between the indistinguishability of paths in interferometric problems and the path-integral formalism in Section 3. Next, we focus on analyzing the EPR argument and the KS contradiction in Section 4. Finally, in Section 5, we draw some conclusions.

2 Quantum and Indistinguishability

The symmetrization postulate (SP) plays a crucial role in standard quantum theory [8]. As one of its consequences, one can derive, for example, the Bose-Einstein and Fermi-Dirac statistics. In the relativistic regime, SP is connected to what is called the spin-statistics theorem. This theorem establishes that integer spin fields are bosons, while half-integer spin fields are fermions [9]. It is important to note that no violation of this principle (or its consequences, such as the Pauli exclusion principle) has ever been empirically observed [10]. We could safely say that SP is one of the major principles of modern physics.

For us, the most important consequence of SP is that quantum systems of the same kind are indistinguishable. In fact, standard quantum mechanics textbooks usually start with indistinguishability and then argue that quantum systems need to be described by either a symmetric or antisymmetric wavefunction. Independently of the interpretation, this means the following. There are physical situations in which it is operationally impossible to specify which quantum system is which.

As a concrete example of this, consider two indistinguishable photons entering the Hong-Ou-Mandel setup [2, 11, 12]. In that example, the two photons P_1 and P_2 enter a beam splitter, as indicated in Figure 2. Detectors D_1 and D_2 are placed on the output paths, and statistics are recorded and analyzed on a computer. When a photon enters a beam splitter, it can be transmitted or reflected. Let the event representing photon 1 reflected be symbolized by R_1 and its transmission by T_1 (with an analogous meaning for R_2 and T_2). When the two photons are sent, there are, in principle, four possibilities, as indicated in Figure 2. Denote the reflection-reflection event by $R_1 - R_2$, a transmission-transmission event by $T_1 - T_2$, and so on. If the photons are sent in the indistinguishability regime (their frequencies and polarizations are the same), we have that the alternatives $T_1 - R_2$ and $R_1 - T_2$ end up being indistinguishable. Therefore, due to a phase change by π during reflections, their amplitudes cancel, and an interference pattern should be observed. Ideally, there is no way to tell which alternative took place, and an interference pattern is indeed observed experimentally.

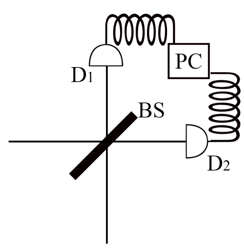

Figure 1: Image of the experimental setup of the HOM effect.

There are plenty of similar examples in relevant areas of quantum physics. Such indistinguishability situations do not have analogs with classical particles. For billiard balls, no matter how similar they are, it is, in principle, always possible to design an operational procedure to distinguish them –and no interference effect will ever be observed. Remarkably, a similar interference effect can be observed with classical fields. But the particle versus field distinction should not lead to confusion: the HOM effect has been observed in massive particles, such as neutrons [13, 14]. On the contrary, if quantum theory is correct, it will be impossible to distinguish operationally quantum systems of the same kind (in certain situations).

We emphasize that there are issues about indistinguishability that are interpretation-dependent. For instance, according to some interpretations, such

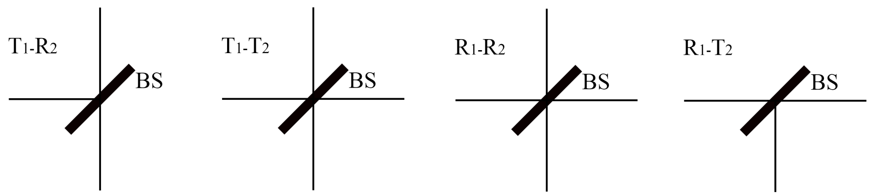

Figure 2: The alternatives $T_1 - T_2$ and $R_1 - R_2$ cannot be discerned if the incoming particles are set in the indistinguishability regime. Accordingly, their amplitudes cancel and an interference pattern is observed. In that case, two incoming photons are always detected in either D_1 or D_2, and it is not possible to tell which photon came from each input port.

as some versions of Bohmian mechanics, particles can be discerned, given that they can be identified by their (hidden) trajectories. But even assuming such hidden-variable theories, one may wonder why identity remains hidden in such a fundamental way. This is an intriguing feature of quantum theory and cannot be considered as a consequence of other features, such as entanglement or superposition. In particular, quantum indistinguishability should not be identified with quantum entanglement. For example, two quantum systems might display genuine quantum indistinguishability but no correlations such that a Bell-type inequality is violated (see, for example, [15]). Alternatively, one can certainly entangle distinguishable particles (for example, particles with different charges or masses).

In the above sense, indistinguishability is always related to compound quantum systems. Therefore, it is associated with collections of systems of the same kind. But there is yet another sense in which indistinguishability is used in quantum theory. Consider the following statement by L. Mandel:

> "[I]t has also been known since the earliest days of quantum mechanics that coherence is related to the intrinsic indistinguishability of the particle trajectories that give rise to the interference pattern. If a photon detected by photodetector D in Fig. 1 can come from either one of the two sources in Fig. 1, and the two possible paths are indistinguishable, then the probability amplitude for the photon to be detected at D is the sum of the probability amplitudes associated with the two possible paths. The detection probability, which is the square modulus of the probability amplitude, then exhibits interference. Obviously coherence and intrinsic indistinguishability are intimately connected." [16]

Similar assertions can be found in Feynman's Lectures on Physics [17].

What is the meaning of the "intrinsic indistinguishability" of the trajectories? Such a rule for interference is very common in physics. It is even taught in undergraduate courses and can be used as a safe principle for everyday work.

But is it possible to give a rigorous ontological meaning to it? Is it related to the usual notion of the indistinguishability of components? In this work, we take as starting point the thesis that indistinguishability can be taken as a crucial ontological feature of quantum systems and that it can be used and taken into account to understand the different interpretational problems of the theory.

3 Indistinguishably of Paths and processes

Let us consider the following quote from R. P. Feynman's famous *Lectures in Physics*.

> "To repeat, do not add amplitudes for different *final* conditions, where by 'final' we mean at that moment the probability is desired—that is, when the experiment is 'finished.' You do add the amplitudes for the different *indistinguishable* alternatives inside the experiment, before the complete process is finished." [17, p. 3-7]

In this section, we devote ourselves to clarifying the meaning of the use of the word "indistinguishable" in the above quotation (and in Mandel's paper [16], quoted in Section 2 of this work). In such interference scenarios, there must not be any way to physically distinguish between two alternate paths or, more generally, processes. In order to illustrate this, we will first consider an example from Feynman and then the Mach-Zender interferometer.

3.1 A scattering example

It is instructive to review here the scattering example discussed in Feynman's Lectures in Physics ([17, p. 3-9]). He considers two scattering particles that can give place to alternative processes, as shown in Figure 3.1. This example was introduced by Feynman in the following way.

> "The next experiment we will describe is one that shows one of the beautiful consequences of quantum mechanics. It again involves a physical situation in which a thing can happen in *indistinguishable* ways, so that there is an interference of amplitudes—as is always true in such circumstances." [17, p. 3-9]

Feynman's reference to the possibility of something happening in two indistinguishable ways is rather curious. Here, we try to analyze what kind of ontology could give more rigorous content to such a possibility. Two particles are emitted by sources S_1 and S_2, and after interacting, they are detected in D_1 and D_2. If the particles were distinguishable, there would be two clearly distinguishable alternatives: The particle from the source S_1 is detected in D_1 and the particle S_2 is detected in D_2, or the other way around. Different processes could be operationally identified, for example, by testing the particles' charges or masses at the output detectors. Thus, if the particles were taken to be of a different kind, then two distinguishable processes would take place. In that case, the probabilities of each process separately must be added, and no interference is expected.

However, if the particles are of the same kind and also have all their internal degrees of freedom in the same state (for example, they have the same spin states), the two processes would become indistinguishable. In that case, we must add the probability amplitudes, and an interference pattern will be observed. For Bosons (such as Alpha particles) we must add the amplitudes, while for Fermions (such as electrons or protons), we must use an extra minus phase to one of the amplitudes (since the probabilities are inside a squared modulus, it is of no relevance which process has the minus sign).

It is important to note that what is relevant here is not that we are personally able or not to inspect the identity of the particles in the detectors. The mere existence of the possibility of revealing which is which destroys any chance of observing an interference pattern, independently of whether we are actually checking identities or not. This feature of the experiment reveals that this is a deep physical property of quantum systems, which seems to have little to do with our ignorance of which process actually takes place.

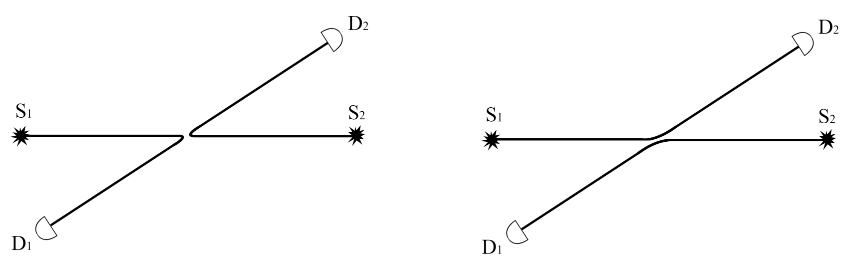

Figure 3: When electrons have the same spin, the above processes cannot be distinguished by any operational means. If that situation is reached, the probability amplitudes must be summed (and not the probabilities directly). Consequently, an interference pattern will be observed.

As noted by Feynmann, this is a very curious effect and lies at the heart of quantum theory. What is the meaning of "indistinguishability" in this example when particles of the same kind are involved? Classically, we can see that there are two different processes. Thus, one might think that even in the quantum case one of the two processes actually took place, and we do not know which one. A Bohmian might explain the interference by appealing to two distinguishable particles guided by a pilot wave. The nonlocal interaction will be responsible for the weird interference effect. But it might also be the case that the reason we cannot operationally distinguish among the alternatives has roots in an alternative ontology. It might well be that, since quantum particles seem to lack individuality, both alternatives become truly indistinguishable when the electrons have the same spins. Only quantum systems could give place to indistinguishable alternatives because they would be nonindividuals (as Schrödinger suggested). An ontology of nonindividuals gives place to truly indistinguishable processes.

3.2 Indistinguishably in the Mach-Zender interferometer

Consider a Mach-Zender interferometer. This is an interesting experiment because, differently from the HOM and scattering examples considered above, it involves a single quantum system on each run. But, as we will show, the assumption of nonindividuality can be seen as playing a crucial role here too. Two (incompatible) experimental contexts are displayed in figure 3.2. In the which-way setup, if a photon is detected in D_1, we can infer that the path 1 was taken. A similar conclusion holds if a photon is detected in D_2. In this experimental context, the two paths can be perfectly distinguished. Therefore, according to Feynman's rule discussed above, no interference is expected. For simplicity and affinity with popular physics jargon, here and in what follows, we will use "paths" to refer to the different alternatives (or processes). But it is important to keep in mind that we do not commit here with an ontology based on particles trajectories. When a second beam splitter BS_2 is placed before the detectors (Figure 3.2 on the right), a detection event in, say, D_1 cannot give us any information about which path the particle took. In that sense, both alternatives become operationally indistinguishable from the point of view of that particular measurement context [1]. The rule is very clear and is an important guide for practical applications of quantum physics. What is the connection between this rule and the traditional notion of quantum indistinguishability?

To see the role of identity, imagine that in one of the arms of the interferometer we place an ideal switch[2] that has the capability to change the identity of the particles that pass through it. That is, if the photon enters arm 1, its type is not changed (i.e., it emerges as a photon), and if it enters arm 2, its type is switched and comes out as an electron. In that case, no interference could be observed since it would be possible to distinguish the particles at the output and find out which way was actually taken. Of course, there is no need to change the particle's identity to realize this experiment: It suffices to change the polarization of the photon in a given path to destroy the interference pattern, as we explain in what follows. Let us add a spin degree of freedom to the quantum system. To fix ideas, think of the paths as modes of the electromagnetic field, and the spin degree of freedom as the photon's polarization. Let the incoming state of the system be $|\psi\rangle_0 = |0\rangle|\uparrow\rangle$ (the system enters in mode $\mathbf{0}$ with polarization $\uparrow$).

[1] Using the standard quantum mechanics' jargon, one can say that the notion of a path is not even well defined when the second beam splitter is put in place. More in line with the spirit of this work, we could say that the processes defined by the which-way context have now become indistinguishable in the new context

[2] Whether this experiment can be actually performed or not is not crucial for our argument. It has the sole purpose of illustrating the idea. A more concrete and realizable example is presented below.

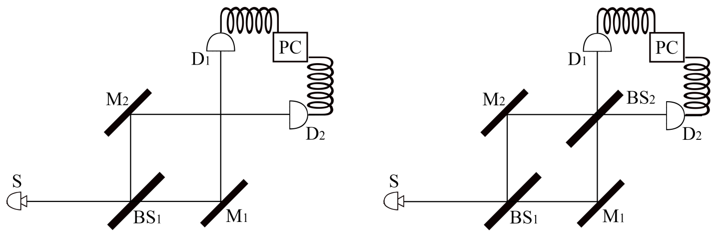

Figure 4: Image of two incompatible experimental setups. On the left, we have a which-way experiment. A photon enters a beam splitter BS_1 and the outcome is detected in D_1 and D_2. Statistics are analyzed on the computer PC. In this setup, the paths can be perfectly distinguished by the detectors. On the right, a second beam splitter precludes the distinction between the two paths. An interference pattern is expected in this context, which gives place to wavelike behavior. This experimental setup is known as a Mach-Zehnder interferometer.

In that case, the state of the system after the first beam splitter BS_1 is given by

$$|\psi_1'\rangle = (BS_1 \otimes I)|0\rangle|\uparrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\uparrow\rangle + i|1\rangle|\uparrow\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)|\uparrow\rangle. \quad (1)$$

Now, suppose that we add a device that changes the polarization of the photon when it enters mode **1**. This new state would be given by

$$|\psi_2'\rangle = P|\psi_1'\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\uparrow\rangle + ie^{i\Delta}|1\rangle|\downarrow\rangle). \quad (2)$$

After applying the second beam splitter, we obtain

$$|\psi_3'\rangle = \frac{1}{2}\left((|0\rangle + i|1\rangle)|\uparrow\rangle + i(1+e^{i\Delta})(i|0\rangle + |1\rangle)|\downarrow\rangle\right). \quad (3)$$

By tracing out the spin degree of freedom, we end up with a mixed state, namely

$$\rho_3 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right). \quad (4)$$

Clearly, no interference can appear under these circumstances, unless we set the polarizer in such a way that the paths become indistinguishable again.

According to the path-integral formalism, the wave function of a system can be written as

$$\psi(x,t) = \frac{1}{Z}\int_{\mathbf{X}(0)=x} \mathcal{D}\mathbf{x} e^{iS[\mathbf{x},\dot{\mathbf{x}}]}\psi_0(\mathbf{X}(\mathbf{t})) \quad (5)$$

where the action S is expressed on a given trajectory as $S = \int dt L(\mathbf{x}(t), \dot{\mathbf{x}}(t))$. The Lagrangian of a free particle can be written as $\mathcal{L} = p\dot{q}$. Therefore, the action takes the form $\mathcal{S} = \int_{path} \mathcal{L}dt = \int p\dot{q}dt = p\Delta q$. For the Mach-Zehnder interferometer, we have four alternative paths. The detection amplitudes in D_1 and D_2 are then given by:

$$D_1 = \frac{1}{2}e^{\frac{i}{\hbar}px_1} - \frac{1}{2}e^{\frac{i}{\hbar}px_2} \tag{6}$$

$$D_1 = \frac{1}{2}e^{\frac{i}{\hbar}px_1} + \frac{1}{2}e^{\frac{i}{\hbar}px_2} \tag{7}$$

The path integral formalism illustrates very well the fact that classically distinguishable trajectories become indistinguishable in the quantum domain. As such, they must be added as amplitudes (and not as probabilities), giving rise to the possibility of interference phenomena.

It is instructive to think about what would have happened to an actual individual, such as a billiard ball, when entering a device that intends to mimic the Mach-Zehnder interferometer. We could envisage a probabilistic mechanism in which, in "beam splitters," the particle is scattered to one part or the other with probability $\frac{1}{2}$. Obviously, at the output detectors, we would observe a $\frac{1}{2}$ probability of detection and no interference at all, independently of whether or not a second beam splitter is placed at the output. Why? According to our discussion, a possible conclusion could be as follows. Only a non-individual can give place to truly indistinguishable alternatives. Only truly indistinguishable alternatives can give way to interference. We can summarize this simply by stating that:

$$\textsf{Noindividuals} \Longleftrightarrow \textsf{Indistinguishability} \Longleftrightarrow \textsf{Interference.}$$

We think the above relationship between an ontology of non-individuals and one of the main features of quantum systems was unexplored in previous works and opens the door for developing a new ontology for quantum theory.

4 A few selected examples from the Foundations of Quantum Mechanics

In this section, we discuss some important examples from the foundations of quantum mechanics that involve, from our point of view, indistinguishability. Some of the examples presented are not commonly considered to have any relationship with indistinguishability. However, we have argued elsewhere [18, 19, 4] that the arguments presented in them are intrinsically or counterfactually based on the ability to assign an identity to the quantum entities involved. As such, we believe that they illustrate the deeper role of indistinguishability in quantum ontology and its connection to identity.

4.1 The Einstein-Podolsky-Rosen Paradox

We begin our discussion with the famous Einstein-Podolsky-Rosen (EPR) "paradox" [6]. Here, we focus on a version of it described by David Bohm in his famous text on quantum mechanics. We start with two spin-1/2 particles, A and B, prepared in the singlet state below.

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B \right]. \tag{8}$$

Here, we use the standard notation that $|+\rangle_i$ means that particle i is in an eigenstate of the spin-z operator $\hat{\sigma}_z$ with eigenvalue $+1$.[3] Similarly for $|-\rangle_i$. For example, if the state is $|+\rangle_A$ and we measure the spin of the particle A in the z direction, we would obtain $+1/2$ with probability 1.

An interesting aspect of an entangled state such as (8) is that the results of the measurements for each particle are highly correlated. For example, take the observable $\hat{\sigma}_z^A \hat{\sigma}_z^B$, corresponding to the simultaneous spin measurement in the direction z for particles A and B. We obtain the following if we compute the expected value of $\hat{\sigma}_z^A \hat{\sigma}_z^B$.

$$E(\hat{\sigma}_z^A \hat{\sigma}_z^B) = \langle\psi|\hat{\sigma}_z^A \hat{\sigma}_z^B|\psi\rangle = -1. \tag{9}$$

This result means that spins are perfectly anti-correlated; if we measure A with spin ± 1, we will measure B with spin ∓ 1 with probability one. The peculiar features of the singlet state 8 are such that the same property would hold if, instead of choosing z, we had chosen any other spatial direction.

This perfect correlation in all possible directions has an interesting consequence. Because A and B are two particles, it is possible to design an experiment in which the state (8) is such that each particle is sent in opposite directions. If this happens, after some time, we may find particle A in Alice's lab and particle B in Bob's lab, both very far apart from each other (as far apart as we may want). This means that if Alice performed an experiment to measure the particle's spin in the z direction (or any other direction), she would know for sure what Bob's result would be if he were to also measure spin in the direction z (or any other direction).

Because Alice and Bob's laboratories are far apart, Einstein, Podolsky, and Rosen (EPR) argued that Alice's measurements cannot, in any way, influence the results of Bob's measurements [6]. Thus, according to EPR, a measurement by Alice allows us to know the results of Bob's measurements without disturbing its system, even if Bob does not perform such a measurement. Therefore, it follows that there should be an element of reality related to the spin value obtained. In other words, according to EPR, the spin of Bob's particle has a definite value before measurement. Also, given that Alice is free to choose the spin of her system in any direction, applying a similar reasoning line, one concludes that the spin of Bob's particles is defined in any possible direction. Given that, in the standard interpretation, a measurement does not simply reveal the value of an observable, EPR claimed that this entangled state shows that quantum mechanics is incomplete.

As is widely discussed, EPR's argument relies on some metaphysical assumptions. First, it requires locality, that is, the impossibility that a measurement

[3] For simplicity, we use units where $\hbar/2 = 1$.

by Alice instantaneously affects a measurement by Bob. This requirement was incorrectly viewed as necessary to ensure that quantum mechanics is consistent with special relativity. Second, based on an argument from nonlocality, EPR calls for the assumption that measurements reveal "elements of reality" of a quantum system. This assumption leads EPR to conclude that quantum mechanics is an incomplete theory. To complete quantum mechanics, the argument goes, one must supplement it by introducing (local) hidden variables. As we shall see in the next section, EPR's criteria of realism can be written down in terms of the existence of joint probability distributions, as implicitly done by John Bell.

However, here we point out an important and often forgotten assumption. It is well known that the EPR argument uses counterfactual reasoning. It assumes that if Alice measured spin $+1$ in direction $\hat{z}$, Bob would have obtained -1 should he have measured spin in the same direction. The same would follow for any other direction, say $\hat{X}$. However, since it is impossible to measure the spin in two different directions simultaneously, EPR needs to assume that the spin values are counterfactually defined. That is, EPR needs to consider that Alice's particle can be viewed in contexts $\hat{z}$ and $\hat{x}$, even if these two contexts cannot co-exist: They necessarily belong to different (possible) worlds. From a purely logical standpoint, the particle of context $\hat{z}$ needs to be identified with the particle of context $\hat{x}$ in order to claim that Bob's particle has a definite value of spin in both directions. Thus, Alice's particle is assumed to retain its identity among two realities that cannot coexist in the same world. This is a natural assumption in classical physics since classical entities are assumed to be individuals.

However, quantum particles are indistinguishable. As such, they lack identity [4][4]. Therefore, even if we assumed that we could argue in a counterfactual way (something that many physicists, including Bohr, refute), if we accept the indistinguishability of particles as part of the quantum ontology, we could not ensure that a particle in one measurement context is the same as a particle in another measurement context. In the best case, we obtain two similar copies (same mass, same intrinsic spin, and same charge). However, even if they are completely alike in their defining properties, they cannot be identified a priori. An extra metaphysical assumption is needed in order to claim that there are no two entities but only one: One needs to assume that the notion of sameness applies to them. According to the ontology of a nonindividual, we cannot do so. Therefore, we could not conclude that we could know the spin in all possible directions in advance. We may be discussing indistinguishable particles, which are different *solo numero.*

[4] It is perhaps relevant to recall here Schrodinger's words: "... we have ... been compelled to dismiss the idea that ... a particle is an individual entity which retains its 'sameness' forever. Quite the contrary, we are now obliged to assert that the ultimate constituents of matter have no 'sameness' at all". [20]

4.2 Bell's criteria of realism

We now turn to Bell's analysis of the EPR experiment. Because Bell's argument uses probabilities, it is helpful to introduce the language of random variables to discuss it. We start with Kolmogorov's definition of a probability space. A probability space is a triple $(\Omega, \mathcal{F}, p)$, where $\mathcal{F}$ is an algebra[5] over the elements of Ω and p is a function that satisfies the following requirements [21].

1. $p : \mathcal{F} \to [0, 1]$,
2. $p(\Omega) = 1$,
3. $\forall A, B \in \mathcal{F}$ with $A \cap B = \emptyset$, $p(A \cup B) = p(A) + p(B)$.

The set Ω is called *sample space*, and $p(A)$ is the probability of the event $A \in \mathcal{F}$.

Intuitively, the algebra $\mathcal{F}$ over Ω gives us a way to attribute probabilities to logical propositions about events. For example, for any $A \in \mathcal{F}$, it follows that A^c, the complement of A, also exists in $\mathcal{F}$. This complement can be thought of as the negation of A, and it follows from the above axioms that $p(A^c) = 1 - p(A)$, as expected. Similarly, $A \cap B$ and $A \cup B$ correspond to the logical conjunction and disjunction. In other words, by requiring the existence of a measure over the algebra over a sample space, we ensure that such a measure satisfies some basic tenets of rationality. This idea can be stated in a more robust way through Cox's Theorem [22] (see [23] for the extension of Cox's theorem to the non-Boolean setting).

Probabilities tell us the likelihood that an event will occur. In an objective interpretation, this likelihood corresponds to the relative frequencies of the empirical observations. In a subjective interpretation, they represent the modeler's subjective beliefs about possible observable results. For example, we can consider obtaining the likelihood of heads and tails when tossing a coin in two different ways. We can perform an experiment in which we toss this coin N times and then count the relative frequencies of heads n_h/N and tails n_t/N, with $n_h + n_t = N$. This is an objective way of thinking about probabilities. Alternatively, without any tossing or experimental observation, we can examine the coin and conclude that we have no reason to believe that heads are more likely than tails and vice versa. Therefore, subjectively, we can argue that the probabilities should be $p(h) = p(t) = 1/2$.[6] We shall not discuss the differences between subjective and objective interpretations of probabilities in detail. We mention them here because they are relevant to some of the following discussions. The interested reader is referred to Maria Carla Galavotti's wonderful book on interpretations of probabilities [24]. Notice that in the standard formulation of quantum mechanics, probabilities are assumed to have an ontological character in the following sense. The description given by the wave function (or,

[5]Technically, $\mathcal{F}$ should be a σ-algebra, i.e., an algebra closed under countable unions and intersections.

[6]An attentive reader may say that this would be problematic, as the coin may be biased. We would point this reader to discussions about the role of Bayes' Theorem in subjective probabilities (see [24]), but here it suffices to say that this is not a problem.

more generally, by the quantum state) is assumed to be *complete* or, in other words, that quantum probabilities are irreducible. This means that, differently from a classical coin toss (which is ultimately governed by rather complicated deterministic dynamics), the reason we need a probabilistic description in QM is not due to our ignorance of the details of the physics of the atoms but to the fact that there is intrinsic randomness in nature [25]. Each time we run a quantum experiment, the unpredictability of the result is assumed to be due to an objective feature of nature and not to our ignorance about the experimental details.

This is the main working hypothesis behind the standard formulation of quantum theory, and it underlies the discussion between Bohr and EPR. Bohr was an adherent of the idea that quantum probabilities are irreducible (taking sides with Heisenberg, Pauli, and Born). The mechanism cannot be known because, being genuinely random, there is nothing to know. Consequently, it cannot be assumed that the values obtained during measurements represent pre-existing quantities associated with the system under study. With their example of a system whose values are well-defined prior to measurements, EPR suggested that quantum mechanics could be indeed completed, opening the door to a research program aimed at discovering hidden variables and re-obtaining an ignorance interpretation for the quantum probabilities. This is connected with Einstein's famous dictum: "God does not play dice."

Thus, the EPR argument and Bell's elaborations can be considered a major challenge to the standard formulation of QM. Although the empirical violation of Bell's inequalities was considered a victory of Bohr's perspective by many physicists[7], some authors still support the idea that there is a place for an ontology based on nonlocal hidden variables. Among them, Bohmians are perhaps the most popular in the philosophy of physics community. Other authors have argued that the unpredictability that pervades all quantum phenomena might be due to an underlying chaotic (hidden) layer of reality. Contextual hidden-variable models, which have the virtue of being local, are worth mentioning.

If probabilities represent the likelihood of outcomes, how can we use them to model real experiments? The representation of the experimental results is done with the use of random variables. Given a probability space, we can define a random variable as a function $R : \Omega \to O$, where O is the (measurable) space of outcomes. For example, if we were trying to represent the outcome of throwing two dice and then adding them, the sample space Ω would have 36 elements (one for each possible result of throwing two dice), but the set of outcomes O would be $O = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with only 11 members. What is nice about random variables is that, once we define them, i.e., the functions, it is straightforward to compute the probabilities associated with each outcome. Furthermore, random variables provide a mathematical way to represent not only one possible experiment but by selecting a reasonable sample space, multiple experiments, or variables. For example, a single probability space could be

[7]To this respect, it is instructive to see the interview with J. Clauser [26] explaining the motivations of some of the experiments that gave rise to the 2022 Nobel Prize in Physics

constructed to model the properties of height, weight, gender, and age in a given population. We can use physics to model any classical experimental outcome and its inevitable fluctuations due to experimental errors. Therefore, extending the use of random variables to model quantum phenomena is natural. This is what we shall do with the EPR experimental setup.

Each quantum observable defines a random variable when restricted to an empirical context. This can be summarized using the spectral theorem. To illustrate this, let us consider the finite-dimensional case. Let the Hermitian matrix A represent an observable (such as the spin of a particle). Let $\{\lambda_i\}$ be the set of eigenvalues of A and $\{P_i\}$ their corresponding projection operators associated with their eigenspaces. Then, we can write that

$$A = \sum_i \lambda_i P_i. \tag{10}$$

As simple as it appears, the above equation has deep implications. First, notice that the set of projection operators $\{P_i\}$ generates a Boolean algebra $\mathcal{F}_A$[8]. A quantum state ρ will define in a canonical way a classical probability distribution p_ρ in $\mathcal{F}_A$ (as in Kolmogorov's axioms above). Therefore, a quantum observable always defines a classical random variable when we restrict ourselves to a concrete empirical measurement to measure it. Intuitively, this means that a classical probability space can describe any concrete empirical context satisfying Kolmogorov's axioms.

Where is the quantum behavior if classical random variables can naturally represent quantum observables? Quantum characteristics arise when we add multiple contexts. Let us put it in a simple but direct way. For some quantum systems, no joint classical probability distribution allows us to compute all possible quantum correlations as marginals (in the usual way) in all possible contexts. Each time we attempt to combine incompatible quantum contexts into a classical joint probability, we face a contradiction with local realism. And this is essentially what Bell does when deriving his inequalities: He assumed the existence of a joint classical probability distribution.

To illustrate the role that identity plays in Bell's derivation, consider the following observables: $\sigma_z \otimes \sigma_z$ and $\sigma_x \otimes \sigma_x$. Operationally, the former implies that both Alice and Bob measure the spin in the $\hat{z}$ direction, while the latter means that Alice and Bob measure the spin in the direction $\hat{x}$ instead. Because we cannot create an experimental procedure that measures the spin in $\hat{x}$ and $\hat{z}$ simultaneously, each observable requires a whole set of distinct experiments. These different experiments define distinct and incompatible empirical contexts. As such, how can we consider them at the same time? There are three possibilities. The first is to consider two equivalent copies of the quantum system (i.e., equivalent preparations) and instantiate the different measurement contexts in different laboratories. The second is to consider a new preparation of the same

[8]In order to see this, define the canonical operations between projections as follows. Let the intersection "$\cap$" of subspaces be the conjunction "$\wedge$", the direct sum $\oplus$ the disjunction "$\vee$", and the orthogonal complement of a subspace "$\perp$" the negation $\neg$. With these operations between subspaces, the eigenspaces of A generate the Boolean algebra $\mathcal{F}_A$.

system (i.e., to perform different measurements at different moments of time). Finally, the third possibility is to consider the contexts as mere possibilities (without the necessity of actually realizing them). In the first option, we explicitly have two quantum systems with indistinguishable particles. In the second case, we have a quantum system considered at different moments of time. Again, if quantum systems lack identity, how can we identify them over time? The third alternative is perhaps the most interesting from a philosophical point of view. How many quantum systems are considered in this case? One could be tempted to say that only one quantum system is considered in two different situations. In other words, the quantum system considered in context $\sigma_z \otimes \sigma_z$ is *the same* as the one considered in context $\sigma_x \otimes \sigma_x$. But in what sense can we say that two systems are the *same*? What sort of metaphysical principle allows us to make an identification? Let us recall Schrödinger's words:

> "I beg to emphasize this and I beg you to believe it: it is not a question of our being able to ascertain the identity in some instances and not being able to do so in others. It is beyond doubt that the question of 'sameness', of identity, really and truly has no meaning." [27, pp. 121-122]

According to Schrödinger's suggested ontology, we cannot apply the notion of sameness to an elementary particle. Or, at least in a weaker version, we cannot do that in all possible situations. Therefore, we must be careful when asserting that "the system is the same." If Schrodinger is right and the questions about sameness and identity have no meaning, we cannot say, at least in all possible situations, that a system considered at different times is the same. Or, more importantly, we cannot say that the equivalent copies of quantum systems considered in incompatible contexts are the same because identity does not apply to them. We can only say that we obtain equivalent copies indistinguishable from each other.

Although identification through incompatible contexts is natural in classical physics, logic does not allow us to jump so quickly into the quantum domain. We must be careful about the ontology. We need an extra ontological assumption if we want to conclude that the obtained copies are indeed the same. In principle, experimental situations $\sigma_z \otimes \sigma_z$ and $\sigma_x \otimes \sigma_x$ inhabit *different possible worlds.*[9] Furthermore, these worlds are *necessarily* different since the experiments are incompatible. One needs a strong metaphysical assumption about the identity of the particles involved if one wants to identify them among these alternative worlds. This is the case in an ontology based on individuals, such as Bohmian mechanics. However, at this point, the ontological weight of the individuality assumption should be clear. One needs to assume that, when considering different (incompatible) contexts, the system considered retains its identity for the existence of a joint probability distribution.

To our knowledge, the role of identity as an ontological assumption was not

[9] Here, we emphasize that we mean possible worlds, not actual, as is the case in the Many Worlds Interpretation of quantum mechanics.

considered in the philosophy of physics literature[10]. This is a crucial assumption, given that if quantum systems are considered genuine nonindividuals, we cannot conclude that the systems from the different alternative possible worlds are the same. In the best case, they are indistinguishable. The impossibility of identifying them a priori reveals that assuming an ontology based on individuals is crucial to grant the existence of a joint probability distribution. From an ontology of nonindividuals, one can only conclude contextual hidden variable models; as is well known, this is not enough to derive Bell inequalities. It is an open philosophical problem to determine to which point an ontology based on nonindividuals necessarily leads to a nonlocal interpretation (such as Bohm's). Here, it suffices to conclude that non-individuality fits perfectly with the contextual behavior of quantum systems. Therefore, as a result of the discussions in this section, we could conjecture the following.

$$\mathsf{Indistinguishability} \implies \mathsf{Contextuality}.$$

4.3 The assumptions underlying the Kochen-Specker theorem

Now, we analyze the Kochen-Specker contradiction from the point of view of the indistinguishability assumption. As will be clear from our analysis, one of the hypotheses that leads to the contradiction is that quantum systems obey the classical rules of identity. In short, the KS theorem assumes that a quantum system retains its identity among the different contexts in which it is considered.

To explain the argument, let us consider an analogy. At some point in life, we may wonder how things might have been if we had chosen a different career. We can imagine an alternate world in which we are not physicists but lawyers. In this world, as physicists, we are deeply interested in quantum foundations. What about our "copy" in the alternate possible world? Can we infer something about their interest in the interpretational problems of quantum theory? It is reasonable to assume that very little can be inferred. Unless we assume that our passion for quantum physics is rooted in our genes or some structural feature of our brain, little can be said about that. We can even state that we are talking about different people. Thus, why should we expect a similar behavior or set of interests? To connect our current selves with those of the alternate realities, we need to make a strong assumption that ensures that we will retain properties in completely different contexts. In particular, we must retain, to a great extent, our identity to arrive at the conclusion that we will behave in the same way.

The situation of quantum systems is similar to the discussion above in the following sense. Two quantum contexts cannot be realized simultaneously: They exist only as possibilities. Thus, the same quantum system cannot be considered in two different incompatible contexts simultaneously. If the system is considered in one context, complementary contexts are necessarily counterfactual; they belong to alternate worlds. Thus, the question arises: Can we identify a quantum system considered in one context with an equivalent one but viewed in

[10]The one exception being our own work in [4], which is forthcoming.

a different context? As we shall see, the answer is ”no” if we are talking about nonindividuals.

Suppose that, when considered in the context C_1, we assign the property P_1 to a quantum system. If we now want to make the assumption that P_1 is retained when considering the system in the complementary context C_2, we need to make a very strong assumption about its identity. Namely, we need to assume that, since it is the same system, it will retain the property in the new context. However, under the assumption that quantum systems are nonindividuals, the rules of classical identity do not apply to them. Therefore, talking about the same system is impossible when considering a different context. In the new context, we obtain a new instantiation of an object of the same class, which is, in the best case, indistinguishable from the one of the previous one (the same mass, the same charge, and the same spin, but in a different context). Thus, the attribution of properties can only be made with regard to a particular measurement context since, for nonindividuals, we cannot even claim that we are speaking of the same entity when we consider different contexts. Under the nonindividuals assumption, there is no logical rule that allows us to deduce that only one individual is the bearer of all the possible properties at once. Therefore, there is no way to reach the global (noncontextual) attribution of properties that leads to the KS contradiction.

5 Conclusions

In this work, we have discussed several interpretational problems of quantum theory considering the possibility that quantum systems belong to the ontological category of *nonindividuals*. In the vast majority of philosophy of physics works, it is implicitly assumed that quantum systems are, in fact, individuals in the sense that they can be labeled, identified, and reidentified in different contexts or situations. Put in logical terms, it is implicitly assumed that the entities involved obey the classical laws of identity. Here, we have stressed that the notions of *identity* and *individual* must be considered essential to define the characteristics of the assumed ontology. Recognizing this implies that one could consider different examples of ontologies in which those features do not hold (or that hold partially). As quantum systems can enter situations where they cannot be operationally discerned, exploring the idea that they lack individuality is natural, as Schrödinger suggested. But, as we have shown here, the assumption of nonindividuality right from the start is game-changing since many of the arguments that give place to the so-called quantum paradoxes tacitly use the objects' identities. When this assumption is relaxed, the arguments do not follow.

We have presented arguments that suggest that ontological indistinguishability is a strong notion that pertains not only to the subsystems of a compound system. Even for a single quantum system, we saw that the assumption of it being a nonindividual can give place to genuinely (and operationally) indistinguishable physical processes. A classical entity, such as a billiard ball, is an individual, and, as such, it cannot generate operationally indistinguishable processes. As the indistinguishability of processes (such as paths in an interferometer) is known to be one of the crucial conditions for quantum interference, our arguments suggest a strong link between nonindividuality and the quantum superposition principle.

More interestingly, there also seems to be a strong link between contextuality and the failure of the classical laws of identity. A plain individual can be operationally distinguished from others of the same kind in every situation. It can be identified and reidentified, be it in different moments of time or different contexts. In contrast, quantum systems seem to lack these features: There are situations in which there is no operationally coherent way to identify and reidentify them. This feature strongly suggests that the classical laws of identity do not hold for quantum systems. However, if that is true, it has strong implications for contextuality. Given that no single world can contain two incompatible measurement contexts, a diversity or plurality is necessarily associated with a quantum system when considered in different contexts. When we conceive of a given entity in different (incompatible) contexts, classical logic only allows us to conclude that we have several copies of it, given that the contexts belong necessarily to different possible worlds. If one wants to suppress diversity and obtain a single entity —*e pluribus unum*—, a strong metaphysical principle of identity must be postulated. From the discussions presented here, it follows that this identification can be carried out only for entities that obey the classical laws of identity (such as billiard balls). For quantum systems, on the other hand, if we assume the hypothesis of nonindividuality, there is no way to conclude that identification can be performed. Only indistinguishable copies are obtained when we consider a quantum system in different contexts. A similar consideration applies to a quantum system at different moments of time or two equivalent preparation procedures (that generate two equivalent copies of the "same" system).

We believe that the arguments presented in this work open the door to further philosophical investigation of questions that had never been considered. In particular, it remains to understand in more detail what happens with the so-called quantum nonlocality in an ontology of nonindividuals[11]. Another interesting question is this: At what point can ontological indistinguishability be used to isolate quantum theory from all possible no-signal generalized probabilistic theories? We hope to address these questions in future work.

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[11] For an analysis based on bundles of properties, see [28, 29].

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