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Photons, Billiards and Chaos

P. Suppes, J. Acacio de Barros

Stanford University, USA

1 Introduction

In this paper we continue the foundational study of photons as particles without wave properties (Suppes and de Barros (1994a), Suppes and de Barros (1994b)). In the earlier work we assumed: (*i*) Photons are emitted by harmonically oscillating sources. (*ii*) They have definite trajectories. (*iii*) They have a probability of being scattered at or absorbed in the near presence of matter. (*iv*) Detectors, like sources , are periodic. (*v*) Photons have positive and negative states which locally interfere, i. e. annihilate each other, when being absorbed. In this framework we are able to derive standard diffraction and interference results. We thereby eliminate in this approach wave-particle duality for photons, and give nonparadoxical answers to standard questions about interference. For example, in the two-slit experiment each photon goes through only one slit.

In the earlier work we did not construct a stochastically complete model of the monochromatic harmonically oscillatory point source, but only assumed an expectation density for the emission of positive and negative photons, namely $n_{\pm}(t) = \frac{A}{2}(1 \pm cos\omega t)$ with $t > 0$. In the present paper we derive this equation (see (8) in Section 1) as the expectation density in free space-time from a probabilistic model of a two-level atom as source, which easily generalizes to N atoms. We also can go further and derive from the model the cross-correlation of two arbitrary space-time points, but we do not include that calculation in this paper.

In Section 2 we look at photons as particles which have, in certain special environments, ergodic motion. In particular, we study the way in which photons having definite trajectories can move in ergodic fashion like billiard balls on a rectangular table with a convex obstacle in the middle. Such billiards are called Sinai billiards after the Russian mathematician Ya.G.Sinai. Their ergodic motion is strongly chaotic.

Finally, in Section 3 we examine the isomorphism of deterministic and stochastic models of photon ergodic motions. Here we use important results of D.S.Ornstein and his colleagues on the indistinguishability of these two kinds of models of ergodic behavior.

The positive and negative photons we introduce can well be thought of as virtual photons, for in a detector they locally interfere with each other, and only the excess of one or the other kind is observable (for further details see Suppes and de Barros (1994b)).

2 Two-Level Atom as Source

We have several processes at the source, which we initially treat as a single atom. In this version we begin by making time discrete, with the time between the beginning of successive trials on the order of the optical period, 10^{-15} s.

Process I. Pure Periodic Process. On an odd trial a photon in the positive state may be emitted or absorbed, and on an even trial a photon in the negative state may be emitted or absorbed. This process is defined by the function

$$f_{\pm}(n) = n \bmod 2, \tag{1}$$

where n is the trial number. Intuitively, we use (1) to make the probability zero of an atom emitting or absorbing a negative photon on trial n if n is odd, and probability zero for a positive photon if n is even. This is our periodicity.

If, on an even-numbered trial, the atom is in the excited state, which we label 1, at the beginning of the trial, then there is a positive probability, but not in general probability 1, of emitting a negative photon, and similarly on odd trials for a positive photon. Correspondingly we use 0 as the label for the ground state

Process II. Exponential Waiting Times. We use a discrete Markov chain in the two states 0 and 1 to give us in the mean the geometric distribution of waiting for absorption or emission, but with different parameters. The geometric distribution is, of course, the discrete analogue of the exponential distribution in continuous time. The transition matrix is:

$$\begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 - c_1 & c_1 \\ 0 & c_0 & 1 - c_0 \end{array} \tag{2}$$

Thus, c_0 is the probability of absorbing a positive or negative photon when in the ground state at the beginning of a trial. In our simple model meant for low-energy experiments, e.g., those dealing with the optical part of the electromagnetic spectrum, we exclude the possibility of multiple-photon absorption or emission on a given trial. The parameter c_1 is just the probability of emitting a positive or negative photon when in the excited state at the beginning of a trial.

Processes I & II together. The description just given of absorption and emission of photons is for Process II alone. Combined with the periodicity of Process I, we can write a single matrix, but one that depends on whether the trial number is odd or even:

$$\begin{array}{c|cc} & 1 & 0 \\ \hline 1 & 1 - c_1 f_{\pm}(n) & c_1 f_{\pm}(n) \\ 0 & c_0 f_{\pm}(n) & 1 - c_0 f_{\pm}(n) \end{array} \tag{3}$$

where $f_+(n) = 1$ if n is odd and 0 if n is even, and contrariwise for $f_-(n)$.

Asymptotic Distribution of States. The Markov chain characterized by (2), for $c_0, c_1 \neq 0$ is obviously ergodic, i.e., there exists a unique asymptotic stationary distribution independent of the initial probability of being in either state.

For computing this distribution we can ignore the distinction between the even and odd numbered trials, as expanded in (3), and consider only the process characterized by (2).

The asymptotic distribution is just obtained from the recursion:

$$p_0 = c_1 p_1 + (1 - c_0) p_0. \tag{4}$$

Solving,

$$p_0 = \frac{c_1}{c_0 + c_1} \tag{5}$$

and

$$p_1 = \frac{c_0}{c_0 + c_1}. \tag{6}$$

So, asymptotically, for N atoms, the expected number in the ground state is $\frac{c_1 N}{c_0 + c_1}$.

Properties of Photons. Omitting here polarization phenomena, a photon is a 3-tuple $(\omega, c, \pm)$, where ω is the frequency of oscillation of the source, c is the velocity, and $\pm$ are the two possible states already discussed.

Process III. Direction of Emission. We assume statistical independence from trial to trial in the direction of emission of photons by a single atom. We also assume that the probability of direction of emission is spherically symmetrical around the point source. For this analysis we further restrict ourselves to two dimensions and a scalar field, as is common in the study, e.g., of optical interference in the two-slit experiment, or the "billiards" case discussed in the next section.

Periodic Properties of $f_\pm(n)$. Various properties are needed.

(i) If φ is even, $f_\pm(n + \varphi) = f_\pm(n)$.

(ii) If n is even, $f_\pm(n + \varphi) = f_\pm(\varphi)$,

(iii) $f_\pm(n + \varphi) = f_\pm(n + 1) f_\pm(\varphi) + f_\pm(n) f_\pm(\varphi + 1)$.

This is easy to prove by considering the four cases: n is odd or even, and so is φ.

For $r \neq 0$,

$$P(\text{photon being at } (r, \theta, t) | \text{ emission at } t') = \frac{1}{4\pi r} \delta(t - t' - \tfrac{r}{c}).$$

We now compute the unconditional probability of emission at t', where gr.st. is the event of being in the ground state.

$$\begin{aligned} P(\pm \text{ emission at } t') &= \sum_{t''<t'} P(\pm \text{ emission at } t' | \text{ absorption at } t'') \\ &\quad \cdot P(\text{absorption at } t'') \\ &= \sum_{t''<t'} \sum_{t'''<t''} P(\pm \text{ emission at } t' | \text{absorption at } t'') \\ &\quad \cdot P(\text{absorption at } t'' | \text{gr.st. at } t''')\ P(\text{gr.st. at } t''') \\ &= \sum \sum c_1 (1 - c_1)^{t' - t''} f_\pm(t') c_0 (1 - c_0)^{t'' - t'''} \frac{c_1}{c_0 + c_1}. \end{aligned} \tag{7}$$

The continuous analogue of (7) is obvious, where β_0 corresponds to c_0 and β_1 to c_1.

$$P(\pm \text{ emission at } t') =$$

$$= \int_{-\infty}^{t'}\int_{-\infty}^{t''} \beta_1 e^{-\beta_1(t'-t'')}(\frac{1}{2} \pm \frac{1}{2}\cos\omega t')\beta_0 e^{-\beta_0(t''-t''')}\frac{\beta_1}{\beta_0+\beta_1} \cdot dt''' dt''$$

$$= \frac{\beta_1}{\beta_0+\beta_1}(\frac{1}{2} \pm \frac{1}{2}\cos\omega t')\left[e^{-\beta_1(t'-t'')}\Big|_{-\infty}^{t'} e^{-\beta_0(t''-t''')}\Big|_{-\infty}^{t''}\right]$$

$$= \frac{\beta_1}{\beta_0+\beta_1}(\frac{1}{2} \pm \frac{1}{2}\cos\omega t'). \quad (8)$$

Let us abbreviate the space-time position as $\imath = (r_\imath, \theta_\imath, t_\imath)$. Then the random variables for positive and negative photons are defined as:

$$X_\imath(\pm) = \begin{cases} 1 \text{ if there is a } \pm \text{ photon (from the single atom) at } (r_\imath, \theta_\imath, t_\imath), \\ 0 \text{ otherwise.} \end{cases} \quad (9)$$

Then, since $t_\imath - t'_\imath = \frac{r_\imath}{c}$, we have

$$E(X_\imath(\pm)) = \frac{1}{2\pi r_\imath}\frac{\beta_1}{\beta_0+\beta_1}(\frac{1}{2} \pm \frac{1}{2}\cos\omega(t_\imath - \frac{r_\imath}{c})), \quad (10)$$

which is the form of the expectation density, $h_\pm = E(X_\imath(\pm))$, for positive and negative photons derived in Suppes and de Barros (1994b), but here for a single-atom source.

3 Photons as Billiards

We now move to the study of photons as particles executing ergodic motions. Let us begin with a rectangular box that has reflecting sides. We assume the classical law of reflection, that is, the angle of reflection equals the angle of incidence. An ideal laser could execute the periodic motion shown in Figure 1 in such a box. In fact, in classical mechanics it is always in terms of such motions that billiards are used as standard examples of mechanical systems. It is only in the modern study of billiards that matters become much more complicated. Intuitively it is easy to describe how to get such additional complication. We add a convex obstacle to the rectangular box with reflecting sides as shown in Figure 2.

Now the path of the photon emitted by an ideal laser executes the motion of a photon as a Sinai billiard. Sinai and other investigators have studied very thoroughly the mechanical motion of a point particle in such a rectangular box with a convex obstacle and with the collisions of the particle observing the classical law of reflection as well as the law of perfect elasticity. In the case

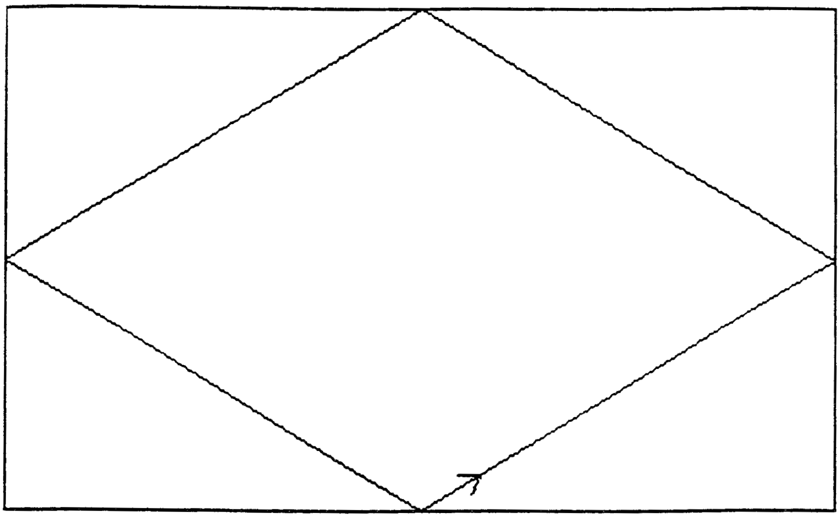

Fig. 1. Periodic photon billiard motion

of photons we drop the concept of elasticity but have the reflection take place without loss of energy.

It is worthwhile to analyze the law of reflection more carefully in our theory. We give a semiclassical derivation in that we assume the reflecting walls are continuous perfect conductors in the sense of classical electromagnetic theory. The boundary condition for a scalar field is that it be zero at the conductor surface. We can show how we obtain this result most clearly by returning to the temporally discrete model introduced at the beginning of Section 1. For a perfect conductor we change the basic assumptions as follows. We modify the transition matrix (3) to reflect the assumption that in reflection a photon changes its state from positive to negative and vice versa.

This single change for perfect conductors enables us to show that the defined scalar electric field at the reflecting surface is zero. For a single photon, we have at once from the new transition matrix below that at a point of the surface, $h_+ = h_-$ and therefore $\mathcal{E} = 0$. We use here the definition of the field $\mathcal{E}$ given in Suppes and de Barros (1994b), i.e., $\mathcal{E} = \frac{\mathcal{E}_0(h_+ - h_-)}{\sqrt{h_+ + h_-}}$. We thus replace (3) by the

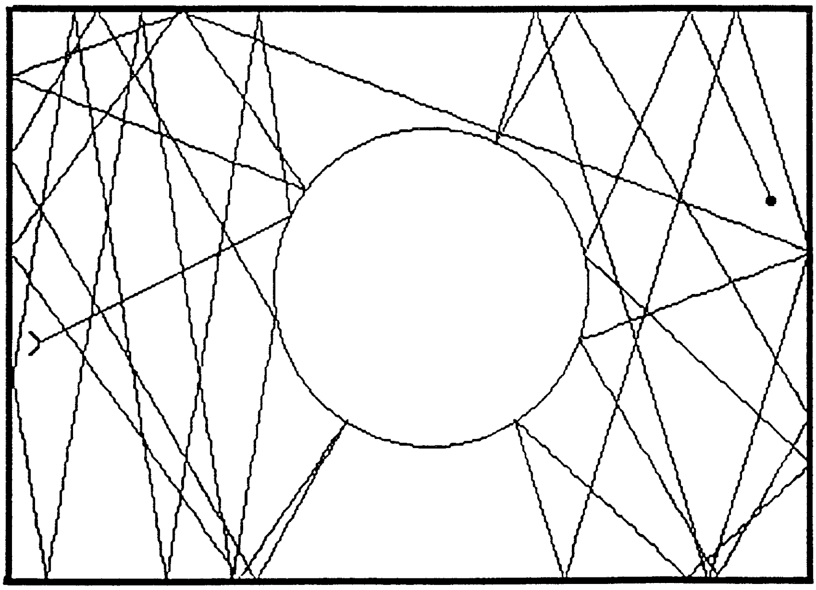

Fig. 2. Photon billiard motion with a convex obstacle

matrix

$$
\begin{array}{c|lll}
 & 1- & 1+ & 0 \\
\hline
1- & 1 - c_1 f_+(n) & 0 & c_1 f_+(n) \\
1+ & 0 & 1 - c_1 f_-(n) & c_1 f_-(n) \\
0 & c_0 f_-(n) & c_0 f_+(n) & 1 - c_0
\end{array}
\qquad (11)
$$

It follows from fundamental results of Sinai (1970) that the following theorem can be proved.

Theorem 1 *The motion of a photon as a Sinai billiard, as shown in Figure 2, is ergodic.*

We can see this even more clearly by showing the picture of a simulation of the motion of a photon as a Sinai billiard. It is intuitively clear that we get then the following corollary from the ergodic motion.

Corollary 1 *The motion of a photon as a Sinai billiard is strongly chaotic.*

We say more about this chaos later, but remember the chaos is derived here from our conception of photons as point particles executing linear trajectories between points of reflection.

Measurement of photons. The chaotic motion of photons is not a topic usually discussed in the many physical discussions of chaos. Why is this? The answer is that in terms of observation using, for example, photodetectors, we can only measure the intensity of a light source averaged over time. It takes about 10^{-9} seconds for an atom to absorb a photon. In contrast, a single period of an optical source is about 10^{-15} seconds. Thus a photon that is emitted by an optical source takes in terms the period of the source on average about 6 orders of magnitude to be absorbed. This averaging process means that there is little hope of observing directly the chaotic motion of an individual photon. These remarks about averaging apply to quantum mechanics and classical electromagnetic theory of optical phenomena, as well as to the probabilistic atom model developed in the previous section. The average intensities predicted by quantum mechanics, by semi-classical application of classical electromagnetic theory, or by the kinds of probabilistic computations developed in the previous section are all average intensities that wipe out in the measurement process any evidence of chaos in the motion of an individual photon.

This means that our straightforward "free particle" theory of photons leads directly to a theory of chaos for photons, but the chaos is not observable by standard means.

4 Deterministic and Stochastic Models

It is widespread folklore in discussions of chaos by physicists that most important physical examples of chaos are deterministic. On the other hand, there is a variety of evidence, especially mathematical arguments, that associated with chaos, particularly in the strongest chaotic examples, are phenomena that can only be regarded as genuinely random or stochastic in nature. It would be easy to argue that one has got to choose either the deterministic or stochastic view of phenomena, and at least for a given set of cases, it is not possible to move back and forth in a coherent fashion. It is this view, also perhaps part of the folklore, that we want to argue very much against in the present discussion. We shall refer occasionally to our work on photons, but we will be depending much more on general ideas from ergodic theory and in particular on the strong kind of isomorphism theorems proved by Donald Ornstein and his colleagues. Before we turn to the details, there are one or two other points we want to discuss in a very intuitive fashion. For example, if we take a billiard model of the photon, or if you want, a mechanical particle, and we consider the deterministic model in the case of an ergodic motion, that is, one, for example, where there is a convex obstacle as shown in Figure 2, then there is an empirically indistinguishable stochastic model. The response to this isomorphism might be, "Well yes, but for the case of ergodic motion where the convex object is present we should choose either the simple Newtonian model or in the case of the photon, the simple deterministic reflection model, really from geometrical optics". Because this Newtonian or geometrical optical model works so well in the nonergodic periodic case when there is no convex object, it is natural to say that it is not a real choice between the deterministic or stochastic models. Because of its generalizability the choice obviously is the deterministic model.

But this argument can run too far and into trouble when we turn to a wider set of cases. On the same line we would be pushed to argue that the only kind of complete physical model for quantum mechanics must be a deterministic one, for example the kind advocated by Bohm, but the evidence once we turn to quantum mechanical phenomena seem far from persuasive for selecting as the unique intuitively correct model the deterministic one. Here there is much to be said for choosing the stochastic model, which is much closer in spirit to the standard interpretation of classical quantum mechanics. Our point, without going into details at this juncture, is that whether we intuitively believe the model should be deterministic or stochastic will vary with the particular physical phenomena we are considering. What is fundamental is that independent of this variation of choice of examples or experiments is that when we do have chaotic phenomena, especially when we have ergodic phenomena, then we are in a position to choose either a deterministic or stochastic model. When such a choice between different models has occurred previously in physics—and it has occured repeatedly in a variety of examples, such as free choice of a frame of reference in Galilean relativity, or choice between the Heisenberg or Schroedinger representation in quantum mechanics—, the natural move is toward a more abstract concept of invariance. What is especially interesting about the empirical indistinguishability and the resulting abstract invariance in the present case, is that at the mathematical level the different kinds of models are inconsistent, that is, the assumption of both the deterministic and the stochastic model leads to a contradiction when fully spelled out. On the other hand, it leads to no contradiction at the level of observations, as we shall see in an important class of ergodic cases.

Entropy and measure-theoretic isomorphism. In order to look at the entropy of appropriate processes, we begin with some of the simplest examples. Without much thought it is clear that the simplest example is a Bernoulli process with a finite number of alternatives and discrete trials. We shall call a finite discrete Bernoulli process any stochastic process with the following features. It is a probability space with a transformation namely a quadruple $(\Omega, \Im, \mu, T)$ satisfying the following assumptions: There is a finite set S and a probability measure p_i on S such that $\sum_{i \in S} p_i = 1$ and where Z is the set of integers, $\Omega = S^Z$, $\Im$ is the product σ-algebra on Ω, μ is equal to the product measure on Ω, and T is a left shift on Ω which means that if x, y are in Ω and for every n $y_{n-1} = x_n$, then $T(x) = y$. We say more about the shift T below. Notice that in this definition x and y are doubly infinite sequences, that is, they are sequences going from $n = -\infty$ to $n = \infty$ and the product measure guarantees that we have independence from trial to trial. Continuing with our Bernoulli example, and having it in the back of our minds, but not restricted to it, if we have a stochastic process defined in terms of a doubly infinite sequence of random variables, $\ldots, X_{-1}, X_0, X_1, \ldots, X_n, \ldots$ then we define the *entropy* rate as the following limit, if it exists, for a finite sequence of random variables.

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{1}{2n+1} H(X_{-n}, \ldots, X_0, \ldots, X_n) \tag{12}$$

and for the independence that is the strong feature of the Bernoulli process we have at once

$$H(\mathcal{X}) = \lim_{n\to\infty} \frac{H(X_{-n}, \ldots, X_0, \ldots, X_n)}{2n+1} = \lim_{n\to\infty} \frac{2n+1}{2n+1} H(X_0) = H(X_0). \tag{13}$$

By a similar line of argument for a finite-state discrete Markov chain we get for its entropy rate the following expression:

$$H(\mathcal{X}) = -\sum_i p_i \sum_j p_{ij} \log p_{ij}. \tag{14}$$

Note that of course for the Markov process always $\sum_j p_{ij} = 1$.

In the ergodic literature there has been an intense study of how the entropy rate of a process relates to the measure-theoretic isomorphism of processes. (Terminology differs in the literature; what we call entropy rate is often just called entropy, but there are several different but closely related concepts of entropy, and the differences are not just a matter of terminology.) For that purpose we need an explicit definition of isomorphism. Let us first begin with a standard probability space $(\Omega, \Im, P)$, where it is understood that $\Im$ is a σ-additive algebra of subsets of Ω and P is a σ-additive probability measure on $\Im$. We now consider a mapping T from Ω to Ω. We say that T is *measurable* if and only if $A \in \Im \rightarrow T^{-1}A = \{\omega : T\omega \in A\} \in \Im$, and even more important, T is measure preserving, that is, $P(T^{-1}A) = P(A)$. T is *invertible* if the following three conditions hold: (i) T is $1-1$, (ii) $T\Omega = \Omega$, and (iii) If $A \in \Im$ then $TA = \{T\omega : \omega \in A\} \in \Im$. It is the measure preserving shift T introduced above that is important. Intuitively this property corresponds to stationarity of the process—a time shift does not affect the probability laws of the process.

We now characterize isomorphism of two probability spaces on each of which there is given a measure-preserving transformation, whose domain and range need only be subsets of measure one, to avoid uninteresting complications with sets of measure zero that are subsets of Ω or Ω'. Then we say $(\Omega, \Im, P, T)$ is *isomorphic in the measure-theoretic sense* to $(\Omega', \Im', P', T')$ if and only if there exists a function φ: $\Omega_0 \rightarrow \Omega'_0$ where $\Omega_0 \in \Im, \Omega'_0 \in \Im'$, $P(\Omega_0) = P(\Omega'_0) = 1$, and φ satisfies the following conditions:

(i) φ is $1-1$,

(ii) If $A \subset \Omega_0$ & $A' = \varphi A$ then $A \in \Im$ iff $A' \in \Im'$, and if $A \in \Im$

$$P(A) = P'(A'),$$

(iii) $T\Omega_0 \subseteq \Omega_0$ & $T'\Omega'_0 \subseteq \Omega'_0$,

(iv) For any ω in Ω_0

$$\varphi(T\omega) = T'\varphi(\omega).$$

To show how recent fundamental results are about the relation between entropy rate and measure-theoretic isomorphism, it was an open question in the 1950s whether the two finite state discrete Bernoulli processes $B(1/2, 1/2)$ and $B(1/3, 1/3, 1/3)$ are isomorphic. (The notation here should be clear $B(1/2, 1/2)$ means that the probability for the Bernoulli process with two outcomes on each trial is that for each trial the probability of one alternative is 1/2 and of the other 1/2). The following theorem clarified the situation.

Theorem 2 *(Kolmogorov (1958), Kolmogorov (1959) and Sinai (1959)). If two finite-state, discrete Bernoulli or Markov processes have different entropy rates, then they are not isomorphic in the measure-theoretic sense.*

Then the question became whether or not entropy is a complete invariant for measure-theoretic isomorphism. The following theorem was proved a few years later by Ornstein.

Theorem 3 *(Ornstein (1970)). If two finite-state, discrete Bernoulli processes have the same entropy rate then they are isomorphic in the measure-theoretic sense.*

This result was then soon easily extended.

Theorem 4 *Any two irreducible, stationary, finite-state, discrete Markov processes are isomorphic in the measure-theoretic sense if and only if they have the same periodicity and the same entropy rate.*

We then obtain:

Corollary 2 *An irreducible, stationary, finite-state discrete Markov process is isomorphic in the measure-theoretic to a finite-state discrete Bernoulli process of the same entropy rate if and only if the Markov process is aperiodic.*

We can go further in terms of photons and billiards with the concept of measure-theoretic isomorphism. To keep things in the context of finite-state discrete processes, we can form a finite partition of the free surface on the billiard table, as shown in Figure 3. This constitutes a finite partition of the space of possible trajectories for the photon or billiard and we correspondingly make time discrete in terms of movement from one element of the partition to another. With these constructive approximations, the following theorem has been proved:

Theorem 5 *(Gallavotti and Ornstein (1974)). With the discrete approximation of the continuous flow just described above, the discrete deterministic model of the photon or billiard is isomorphic in the measure-theoretic sense to a finite-state discrete Bernoulli process model of the motion of the photon or billiard.*

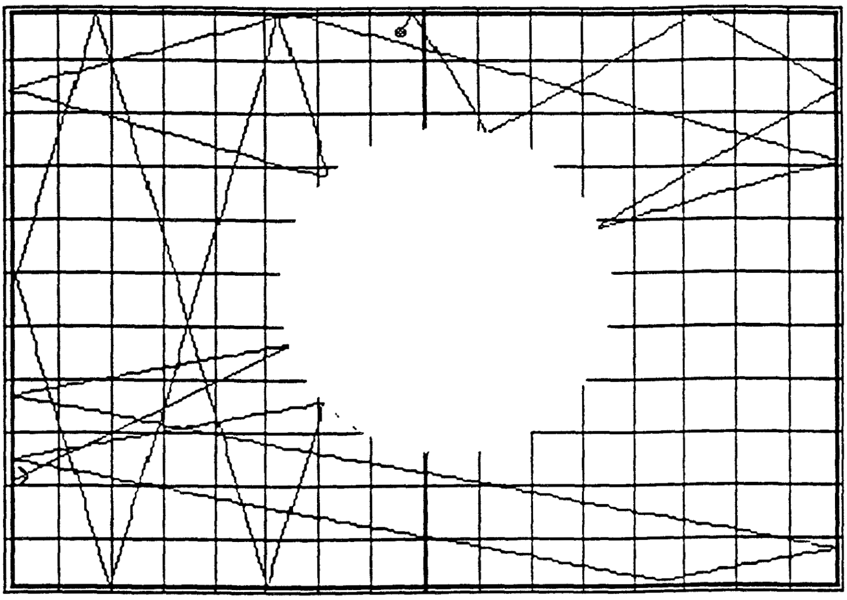

Fig. 3. Finite partition of the billiard table with convex obstacles

; should be noted that instead of this theorem we could have stated a theorem ɔr continuous time and such results are to be found in the paper by Gallavotti nd Ornstein. What the Gallavotti and Ornstein theorem shows is that the iscrete mechanics of billiard balls is in the measure-theoretic sense isomorphic o a discrete Bernoulli analysis of the same phenomena. However, it is to be mphasized that in order to claim that intuitively the two kinds of analysis are ndistinguishable from observation we need stricter concepts.

To show this, we need not even consider something as complicated as the ›illiard example but consider only a first-order Markov process and a Bernoulli ›rocess that have the same entropy rate and therefore are isomorphic in the neasure-theoretic sense, but it is also easy to show by very direct statistical tests vhether a given sample path of any length, which is meant to approximate an nfinite sequence, comes from a Bernoulli process or a first-order Markov process. There is for example a simple chi-square test for distinguishing between the ;wo. It is a test for first-order versus zero-order dependency in the process. The ınalysis is statistical and of course cannot be inferred from a single observation, ɔut the data are usually decisive even for finite sample paths that consist of no nore than 100 or 200 trials.

To spell out the details of this test, let $n_{ij}(t)$ denote the observed number of cases (for several possible runs) in state i at $t-1$ and state j at t. Further, let

$$n_i(t-1) = \sum_j n_{ij}(t), \quad n_{ij} = \sum_t n_{ij}(t), \quad n_i = \sum_j n_{ij}.$$

The Markov character of the sequence of position random variables, or of other sequences of random variables, may be tested directly without recourse to theoretical details of the process. We can test the null hypothesis that the outcomes of trials are statistically independent (zero-order process) against the alternative hypothesis that the process is a first-order Markov chain by computing the sum

$$\mathcal{X}^2 = \sum_{ij} n_i \frac{\left(\frac{n_{ij}}{n_i} - \frac{n_j}{N}\right)^2}{\frac{n_j}{N}},$$

where $n_j = \sum_i n_{ij}$, $N = \sum_{i,j} n_{ij}$, and n_{ij} and n_i are as defined above. Again, $\mathcal{X}^2$ has the usual limiting distribution with $(m-1)^2$ degrees of freedom. (A Bayesian modification of this test is easily given.)

A second null hypothesis is that the process is a first-order Markov chain against the hypothesis that it is a second-order chain. Rejection of the null hypothesis in this case would mean that the position probabilities can be predicted better by observing the two immediately preceding positions rather than simply the single immediately preceding one, and so on for $n+1$st-order vs nth order. Similar chi-square tests can be formulated for stationarity.

Congruence. To obtain a stricter sense of isomorphism it is natural to impose a geometric conditon, especially for a wide variety of physical examples of ergodic systems. Here we follow Ornstein and Weiss (1991). Let $\alpha > 0$ and let $\chi = (\Omega, \Im, P, T)$ and $\chi' = (\Omega', \Im', P', T')$ be two spaces isomorphic under φ in the measure-theoretic sense. Then χ and χ' are *α-congruent* if and only if there is a function g from Ω to a metric space (with d the metric) and a function g' from Ω' to the same metric space such that for any ω in Ω, $d(g(\omega), g'(\varphi(\omega))) < \alpha$ except for a set of measure $< \alpha$.

Intuitively the parameter α reflects our inability to measure physical quantities, inlcuding geometric ones, with infinite accuracy. What is significant is that α-congruence for small α, can be proved for Sinai billiards, and thus photons in a Sinai billiard box. And when α is chosen at the finite limit of our measurement accuracy, the Newtonian mechanical and a Markov process model of a Sinai billiard are observationally indistinguishable, as they are α-congruent.

Stated informally, we then have the fundamental result.

Theorem 6 *Using the discrete approximation described just before Theorem 5, the discrete deterministic model of the photon or billiard is observationally indistinguishable from a finite-state discrete Markov model of the motion of the photon or billiard.*

It is important to note that Theorem 6 is not true if the Markov model is replaced by a Bernoulli model. The observable dependencies discussed above, and for which a chi-square test was stated, rule out the Bernoulli model as a candidate for being α-congruent to the deterministic billiard model.

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