A Proposed Experiment Showing that Classical Fields Can Violate Bell's Inequalities

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Abstract

We show one can use classical fields to modify a quantum optics experiment so that Bell's inequalities will be violated. This happens with continuous random variables that are local, but we need to use the correlation matrix to prove there can be no joint probability distribution of the observables.

Key words: classical fields, Bell's inequalities, quantum optics, correlations

1   Introduction

The issue of the existence of hidden variables for quantum mechanics is almost as old as quantum mechanics itself. However, in 1963 J. S. Bell showed [3, 4] that if one makes some "reasonable" assumptions about the hidden variables, like locality and statistical independence of distant measurements, the correlations for the outcome of measurements for an EPR-like experiment have to satisfy a set of inequalities. In 1982 Alain Aspect and coworkers showed that quantum mechanics violated Bell's inequalities, drawing the conclusion that one cannot have a local realistic theory that would replace quantum mechanics [5, 6].

Because Bell's assumptions were considered equivalent to the existence of an underlying physical reality, it is often said that any classical system satisfies

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Bell's inequalities. In this paper we will show that classical fields *do not* satisfy Bell's inequalities, hence classical fields, e.g. electromagnetic fields, are not Bell-type hidden variables. We do this by showing that a simple experimental setup, suggested by Tan *et al.* [1, 2], can be reinterpreted for classical electromagnetic fields. For this reinterpretation we derive from the classical field properties a violation of Bell's inequalities[3, 4, 7], with, at the same time, locality being preserved in a sense to be made precise.

2   Experimental Setup

The experimental scheme uses two classical coherent sources $\alpha_1(\theta_1)$, with phase θ_1, and $\alpha_2(\theta_2)$, with phase θ_2, and a third source to be studied, $u(\theta)$, with unknown phase. The experimental configuration has two homodyne detections, (D_1, D_2) being one and (D_3, D_4) the other, such that the measurements are sensitive to phase changes in $u(\theta)$. The geometry of the setup is shown in FIG. 1. In FIG. 1 BS1, BS2 and BS3 are beam splitter mirrors that will reflect
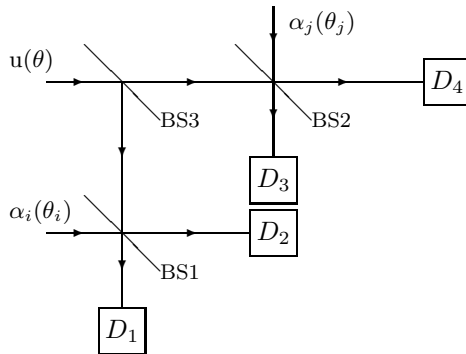

Figure 1: Proposed experimental configuration.

50% of the incident electromagnetic field and let 50% of it pass. When the electromagnetic field is reflected, the mirrors add a phase of $\pi/2$ to the field, while no phase is added when the field passes through BS1, BS2 or BS3. We will look for correlations between the pairs of detectors (D_1, D_2) and (D_3, D_4).

3   Correlation Functions

In this section we compute the correlation functions that violate the inequalities. We first define the random variables in terms of which we derive the Bell-type correlations. On this matter we shall be as explicit as possible. Associated to

the source $u(\theta)$ at D_1 is the random variable $U_1(t)$, whose value at t is just the value of the classical field at D_1, namely,

$$U_1(t) = \frac{1}{4}\beta \cos(\omega t + \theta + \pi/2), \tag{1}$$

where β is the amplitude of the field at the source, θ is the unknown phase and $\pi/2$ is a phase gained when u is reflected at BS3.

Probability enters by using the time average to compute the expectation of $U_1(t)^2$

$$U_1^2 = \langle U_1(t)^2 \rangle = \langle [\frac{1}{4}\beta \cos(\omega t + \theta + \pi/2)]^2 \rangle, \tag{2}$$

which is just the standard intensity, but here we treat it probabilistically. In a similar fashion, associated to the source $\alpha_1(\theta_1)$ at D_1 is the random variable $A_1(t)$,

$$A_1(t) = \frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2) \tag{3}$$

and thus

$$A_1^2 = \langle A_1(t)^2 \rangle = \langle [\frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2)]^2 \rangle. \tag{4}$$

At D_1, the total field is the random variable $F_1(t) = U_1(t) + A_1(t)$. So, the intensity of the total field at D_1 is just the second moment of $F_1(t)$, i.e.,

$$\begin{aligned} I_1(\theta) &= F_1^2 = \langle F_1(t)^2 \rangle = \langle (U_1(t) + A_1(t))^2 \rangle \\ &= \langle U_1(t)^2 \rangle + 2\langle (U_1(t)A_1(t)) \rangle + \langle A_1(t)^2 \rangle, \end{aligned} \tag{5}$$

where we used θ as an argument for I_1 to make it explicit that it depends on θ. We can see that the cross moment in the expression above is the classical interference term.

We can compute I_1 directly from the expression for $U_1(t)$ and $A_1(t)$ in the following way

$$\begin{aligned} I_1(\theta) &= \lim_{T \to \infty} \frac{1}{T} \int_0^T [\frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2) + \\ &\qquad \frac{1}{4}\beta \cos(\omega t + \theta + \pi/2)]^2 dt, \end{aligned} \tag{6}$$

which is

$$I_1(\theta) = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha\beta \cos(\theta - \theta_1) + \frac{1}{8}\alpha^2. \tag{7}$$

In similar fashion, we can compute for the other three detectors,

$$I_2(\theta) = \frac{1}{32}\beta^2 - \frac{1}{8}\alpha\beta \cos(\theta - \theta_1) + \frac{1}{8}\alpha^2, \tag{8}$$

$$I_3(\theta) = \frac{1}{32}\beta^2 - \frac{1}{8}\alpha\beta \sin(\theta - \theta_2) + \frac{1}{8}\alpha^2, \tag{9}$$

and

$$I_4(\theta) = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha\beta\sin(\theta - \theta_2) + \frac{1}{8}\alpha^2. \tag{10}$$

The intensities obtained are conditional on θ. To obtain the unconditional intensities we assume a uniform distribution for θ and integrate the expressions for all possible values of θ. Not only is θ unknown, but the phase would vary randomly for repeated runs of the experiment. If θ were a coherent source with fixed θ, Bell's inequalities would not be violated [8].

The unconditional intensities I_1, I_2, I_3, and I_4 for the detectors D_1, D_2, D_3, and D_4 are

$$I_1 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{11}$$

$$I_2 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{12}$$

$$I_3 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{13}$$

$$I_4 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2. \tag{14}$$

We can see from (11)–(14) that the intensities are the same for all detectors, and are similar to those given by Walls and Milburn [2] in the case of a classical source.

We now start computing the covariance between intensities in the homodyne detectors. The covariance we are interested in is between $(I_1 - I_2)$ and $(I_3 - I_4)$.

$$\mathrm{Cov}(I_1 - I_2, I_3 - I_4) = \frac{1}{2\pi}\int_0^{2\pi}[(I_1(\theta) - I_2(\theta)) \times (I_3(\theta) - I_4(\theta))]d\theta$$

$$- \frac{1}{2\pi}\int_0^{2\pi}(I_1(\theta) - I_2(\theta))d\theta \times \frac{1}{2\pi}\int_0^{2\pi}(I_3(\theta) - I_4(\theta))d\theta. \tag{15}$$

It is straightforward to show from (7)–(10) and (15) that

$$\mathrm{Cov}(I_1 - I_2, I_3 - I_4) = -\frac{1}{32}\beta^2\alpha^2\sin(\theta_1 - \theta_2). \tag{16}$$

In order to compute the correlation we have to know the variance of the random variables $(I_1 - I_2)$ and $(I_3 - I_4)$, which are defined as

$$\mathrm{Var}(I_1 - I_2) = \frac{1}{2\pi}\int_0^{2\pi}(I_1(\theta) - I_2(\theta))^2 d\theta - \left[\frac{1}{2\pi}\int_0^{2\pi}(I_1(\theta) - I_2(\theta))d\theta\right]^2 = \frac{1}{32}\beta^2\alpha^2 \tag{17}$$

and

$$\mathrm{Var}(I_3 - I_4) = \frac{1}{2\pi}\int_0^{2\pi}(I_3(\theta) - I_4(\theta))^2 d\theta - \left[\frac{1}{2\pi}\int_0^{2\pi}(I_3(\theta) - I_4(\theta))d\theta\right]^2 = \frac{1}{32}\beta^2\alpha^2. \tag{18}$$

Finally, we are in a position to compute the correlation between the two random variables $(I_1 - I_2)$ and $(I_3 - I_4)$. This is done in the standard way, by just dividing the covariance by the squareroot of the variances:

$$\rho(I_1 - I_2, I_3 - I_4) = \frac{\mathrm{Cov}(I_1 - I_2, I_3 - I_4)}{\sqrt{\mathrm{Var}(I_1 - I_2)\,\mathrm{Var}(I_3 - I_4)}}, \tag{19}$$

and we have the following expression for the correlation

$$\rho(I_1 - I_2, I_3 - I_4) = -\sin(\theta_1 - \theta_2), \tag{20}$$

which we may rewrite as

$$\rho(\theta_1, \theta_2) = -\sin(\theta_1 - \theta_2). \tag{21}$$

4   Violation of Bell's Inequalities

We are now in a position to show that we can violate Bell's inequalities. We may now choose angles θ_1, θ_2, θ_1', and θ_2' such that we obtain at once, for the four correlations $\rho(\theta_1, \theta_2)$, $\rho(\theta_1, \theta_2')$, $\rho(\theta_1', \theta_2)$ and $\rho(\theta_1', \theta_2')$ a violation of Bell's inequalities in the form due to Clauser, Horne and Shimony [9], by choosing the four angles such that

$$\theta_1 - \theta_2 = \theta_1' - \theta_2' = 60^o, \tag{22}$$

$$\theta_1 - \theta_2' = 30^o, \tag{23}$$

$$\theta_1' - \theta_2 = 90^o. \tag{24}$$

In particular,

$$\rho(\theta_1, \theta_2) - \rho(\theta_1, \theta_2') + \rho(\theta_1', \theta_2) + \rho(\theta_1', \theta_2') =$$
$$-\frac{\sqrt{3}}{2} + \frac{1}{2} - 1 - \frac{\sqrt{3}}{2} \quad < \quad -2. \tag{25}$$

However, in the case of continuous random variables, which is what we have in the present context for intensity, or differences of intensity, failure to satisfy Bell's inequalities in the Clauser, Horne and Shimony form does not imply that there can be no joint distribution of the four random variables compatible with the four given correlations. In fact, it is easy to show that for selected values of the two missing correlations, there does, for this example, exist a joint probability of the four random variables compatible with the four given correlations. What is required in the general case, as opposed to that of discrete ± 1-values, to test for the existence of a joint distribution, when means and correlations are given, is that the eigenvalues of the correlation matrix are all nonnegative. This extends the earlier result of [10] for discrete ± 1 values. Because of the freedom to select arbitrarily the two missing correlations in the Clauser, Horne

and Shimony form of Bell's inequalities, we have not been able to construct an example using $-\sin(\theta_i - \theta_j)$ for the correlations that has at least one negative eigenvalue for all possible values of the missing correlations.

Another possibility is obvious. Bell's original paper [3] used three rather than four random variables, and, put in terms of this letter, he showed that the correlations for the three angle differences derived from θ_1, θ_2, and θ_3 violated the following inequality, necessary for the existence of a joint distribution of three discrete random variables with values ± 1:

$$\rho(\theta_1, \theta_2) + \rho(\theta_1, \theta_3) + \rho(\theta_2, \theta_3) \geq -1. \tag{26}$$

To violate (26) we choose three angles θ_1, θ_2 and θ_3, with

$$\theta_1 = 0, \tag{27}$$

$$\theta_2 = 45^o, \tag{28}$$

$$\theta_3 = 90^o, \tag{29}$$

and, using equation (21), the correlation matrix is

$$\begin{pmatrix} 1 & -\sqrt{2}/2 & -1 \\ -\sqrt{2}/2 & 1 & -\sqrt{2}/2 \\ -1 & -\sqrt{2}/2 & 1 \end{pmatrix}. \tag{30}$$

It is a direct computation to show this matrix has both positive and negative eigenvalues, so that it is not nonnegative definite. The eigenvalues are $(\sqrt{5} + 1)/2$, $(-\sqrt{5} + 1)/2$ and 2. Therefore, there can be no joint probability distribution for the three random variables compatible with the correlations given in (30). From results in [11] and [12] there can then be no hidden variable that factors out the correlations conditionally, i.e., there can be no λ such that for the three random variables $X(\theta_1)$, $Y(\theta_2)$ and $Z(\theta_3)$, we have

$$E(XYZ|\lambda) = E(X|\lambda)E(Y|\lambda)E(Z|\lambda), \tag{31}$$

since there is no joint probability distribution of X, Y and Z compatible with the given correlations. In particular, $\theta = \lambda$ cannot serve as a Bell-type hidden variable for a classical field described by (1).

Finally, we note that even though (26) was violated by the angle values in (27)-(29), this inequality is not a satisfactory general test for existence of a joint distribution, as the following shows. Let $\theta_1 = 0$, $\theta_2 = 30^o$ and $\theta_3 = 45^o$. Then it is easy to check that inequality (26) is violated, but the eigenvalues of the correlation matrix are all nonnegative, and so a joint distribution exists.

5 Measurement and Photon Counts

Because classical field theory is a deterministic theory, our introduction of expectations and probabilities might be questioned. Our response is that the strength of a classical field at a space-time point cannot be measured, as was emphasized long ago by Bohr and Rosenfeld in a famous paper in 1933 [13]. As they pointed out, classical field strength cannot be represented by true point functions, but by average values over space-time regions. This is exactly what we have done in introducing random variables and their expectations. The casual reader might claim that we should do an analysis of coincidence counts with photocounters. This makes no sense in the case of classical fields, where the number of photons arriving at the same time at each detector is incredibly large. What makes sense is not discrete but continuous measurement of intensity.

Despite that, we are going to use the previous result to model discrete photon counts in such a way that they violate Bell's inequalities. For this, we define two new discrete random variables $X = \pm 1$ and $Y = \pm 1$. These random variables correspond to nearly simultaneous correlated counts at the detectors, and are defined in the following way.

$$X = \begin{cases} +1 & \text{if detector } D_1 \text{ triggers a count} \\ -1 & \text{if detector } D_2 \text{ triggers a count} \end{cases} \tag{32}$$

$$Y = \begin{cases} +1 & \text{if detector } D_3 \text{ triggers a count} \\ -1 & \text{if detector } D_4 \text{ triggers a count.} \end{cases} \tag{33}$$

To compute the expectation of X and Y we use the stationarity of the process and do the following. First, let us note that

$$I_1 - I_2 = N_X \cdot P(X = 1) - N_X \cdot P(X = -1), \tag{34}$$

where N_X is the expected total number of photon counts at D_1 and D_2 and $P(X = \pm 1)$ is the probability that the random variable X has values ± 1. The same relation holds for

$$I_3 - I_4 = N_Y \cdot P(Y = 1) - N_Y \cdot P(Y = -1). \tag{35}$$

To simplify we put as a symmetry condition that $N_X = N_Y = N$, i.e., the expected number of photon counts at each homodyne detector is the same. But we know that

$$I_1 + I_2 = N \cdot P(X = 1) + N \cdot P(X = -1) = N, \tag{36}$$

and

$$I_3 + I_4 = N \cdot P(X = 1) + N \cdot P(X = -1) = N. \tag{37}$$

Then we can conclude from equations (7)—(10), assuming maximum visibility, that

$$E_d(X|\theta) = \frac{I_1 - I_2}{I_1 + I_2} = \cos(\theta - \theta_i), \tag{38}$$

$$E_d(Y|\theta) = \frac{I_3 - I_4}{I_3 + I_4} = \sin(\theta - \theta_j), \tag{39}$$

where E_d represents the expected value of the counting random variable. It is clear that if θ is uniformly distributed we have at once:

$$E(X) = E_\theta(E_d(X|\theta)) = 0, \tag{40}$$

$$E(Y) = E_\theta(E_d(X|\theta)) = 0. \tag{41}$$

We can now compute $\text{Cov}(X, Y)$. Note that

$$
\begin{aligned}
\text{Cov}(X, Y) &= E(XY) - E(X)E(Y) \\
&= E_\theta(E_d(XY|\theta)) - E_\theta(E_d(X|\theta))E_\theta(E_d(Y|\theta))
\end{aligned} \tag{42}
$$

and so

$$
\begin{aligned}
\text{Cov}(X, Y) &= \frac{1}{2\pi} \int_0^{2\pi} E_d(XY|\theta)d\theta \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} E_d(X|\theta)d\theta \times \frac{1}{2\pi} \int_0^{2\pi} E_d(Y|\theta)d\theta.
\end{aligned} \tag{43}
$$

In order to compute the covariance, we also use the conditional independence of X and Y given θ, which is our locality condition:

$$E_d(XY|\theta) = E_d(X|\theta)E_d(Y|\theta), \tag{44}$$

because given θ, the expectation of X depends only on θ_i, and of Y only on θ_j. Then, it is easy to see that

$$\rho(X, Y) = \text{Cov}(X, Y) = -\sin(\theta_i - \theta_j). \tag{45}$$

The correlation equals the covariance, since X and Y are discrete ± 1 random variables with zero mean, as shown in (40) and (41), and so $\text{Var}(X) = \text{Var}(Y) = 1$. It follows at once from (45) that for a given set of θ_i's and θ_j's Bell's inequalities are violated.

6   Locality

Much of the discussion involving Bell's theorem is connected to locality. For that reason, we will prove in this section that the scheme presented in this paper is local in one precise sense. We follow [7]. Locality requires the following:

$$E(X|\theta_1, \theta_2, \theta) = E(X|\theta_1, \theta). \tag{46}$$

It is obvious that (46) follows immediately from subtracting (8) from (7), and observing the result does not depend on θ_2, and similarly for the other cases of random variables Y and Z. Equation (46) says simply that whatever is the result of the measurement at one homodyne detector, it must depend only on θ, the hidden variable, and the phase associated to this particular detector, and cannot be influenced by the phase at the other detector.

7 Proposed experiment.

The experiment proposed in [1] supposes a single photon source that is split into the two homodyne detectors. Tan *et al.* also analyze the classical case and get no violation of Bell's inequalities. However, they assume a weak coherent source with randomized phase as the classical analogue of their single photon source. This would be equivalent to having a classical thermal source, where coherence would not be a strong feature. In our experiment we suppose that this source is not only classical, i.e., with high intensity, but also that it is coherent with the phase unobservable and varying randomly on repeated runs. The different source used here, as opposed to that used in [1] implies that the expectations given by (7)–(14) are computed in a different way than in [1]. Here we first integrate with respect to t and them integrate with respect to θ. It is easy to supply a source that would fit our requirements. This would be, for example, a radio source, a microwave, or a laser source, all with unstabilized phases. To realize this experiment, one must also use two additional coherent sources with stable known phases and with the same frequency as the nonstabilized source. If a data table is then built that keeps track of all the measured values on the detectors, we can compute the correlations and see a violation of Bell's inequalities, or do the stronger test using the 3-variable version and the matrix (30).

8 Final Remarks

There are several remarks that we must add in order to clarify some points.

First, when using classical fields the number of photons is overwhelmingly large. For that reason, we would not need to compute any photon count correlation. What we measure is intensity. On the other hand, Bell's inequalities are not enough to show that we do not have a joint probability distribution for classical fields, because they assume a continuous range of values. That is why we computed the correlation matrix, showing that for this case a joint probability distribution does indeed not exist.

Another point is that intensity of classical fields does not satisfy the basic assumption made by Bell, because it can take an infinite range of values; Bell considered spin measurements that can take only two possible values. To show that this does not present any constraint, in Section V we did an analysis of photon counts, which can only take, as in Bell's assumptions, two discrete values.

Finally, the last point. One can argue that if classical fields violate Bell's inequalities, then, since they are classical, Bell's theorem must be wrong, and we must show why it is wrong. We did not show that Bell's theorem is wrong. We just showed that a classical field is not a Bell-type hidden-variable. What is wrong is the preconception that anything classical must satisfy Bell's hypothesis.

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