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Comments on "There Is No Axiomatic System for the Quantum Theory"

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Abstract In a recent paper, Nagata (Int. J. Theor. Phys. 48(12):3532, 2009) claims to derive inconsistencies from quantum mechanics. In this paper, we show that the inconsistencies do not come from quantum mechanics, but from extra assumptions about the reality of observables.

Keywords Realism · Joint observables · Contextualites

Quantum mechanics is one of the best tested theories in modern physics, and yet there is no consensus as to what it means. The reason lies in the fact that, as Feynman eloquently put, "nobody understands quantum mechanics" [2]. This lack of understanding comes from the difficulties to interpret even the simplest of the examples, like the two-slit experiment, in ways that are consistent with the observations and with an underlying ontology that most consider satisfactory (for distinct approaches, see [3–7]). For example, quantum mechanical observables do not allow for standard joint probability measures to be defined [8], and if we assume such probabilities, we derive contradictions [9]. Furthermore, the structure of observables does not satisfy a classical logic, but instead a quantum one [10]. Thus, an area of intense interest in the foundations of quantum mechanics is the search for theories that complete quantum mechanics, such as hidden-variable theories, and give sense to it [6].

In a recent paper, Koji Nagata looked into the possibility that quantum mechanics leads to contradictions [1]. Though, as mentioned in many of the references above, contradictions can be derived depending on the assumptions used, Nagata goes further and claim that "there is a contradiction within the Hilbert space formalism of the quantum theory." He then concludes that no axiomatization exists for quantum mechanics. In this paper, we clarify some assumptions made by Nagata and show that the derived contradictions are not part of the theoretical structure of the theory, but instead are part of metaphysical assumptions about

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the systems. Therefore, the contradictions obtained by Nagata are not an impediment to the axiomatization of the theory, but instead to a specific worldview.

Let us start with Nagata's derivation of a contradiction. In [1], a pure state spin-$\frac{1}{2}$ system on the x–y plane is considered. He then goes on to show that if we compute the quantum mechanical expectation of such state measured in an arbitrary direction $\mathbf{n}$, the expected value $E_{QM} \leq 1$, which implies that $|E_{QM}|_{\max} = 1$. This result is consistent with the fact that, for his choice of units $\hbar/2 = 1$, if the system is prepared in the same direction as $\mathbf{n}$, we always get the same answer as 1. Finally, Nagata shows that if we use a collapsed state and then compute the expectation, $E'_{QM} \leq 2$, implying $|E'_{QM}|_{\max} = 2$ (we changed the notation to E' to avoid confusion with the previous value). Thus, Nagata claims, because $|E_{QM}|_{\max}$ cannot have two different values, we arrive at a contradiction. Before we proceed, we would like to point out that Nagata's inequalities do not by themselves imply a contradiction. For example, the statements $b \leq 1$ and $b \leq 2$ are not contradictory, as $b = 0$ is an example that satisfies them. To prove a contradiction, Nagata would have to construct a system which he could prove not only is less than 2 but is also greater than 1. Though he did not show such proof, it in fact exists. But to make clear where the contradiction comes from, we present it below in a simplified version.

At the core of Nagata's derivation lies an important feature of quantum mechanics, namely that if you do not measure something you cannot assume that it has a value. In other words, assuming values to unmeasured observables leads to contradictions (see [9, 11] and references therein). Let us analyze the case of a spin-$\frac{1}{2}$ system. First, let us see what quantum mechanics can tell us about this system. If we want to observe its spin in a given direction $\mathbf{m}$, the associated observable is $\hat{O}_{\mathbf{m}} \equiv \mathbf{m} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is a vector in $\mathbb{R}^3$ with the Pauli matrices as components, i.e. $\boldsymbol{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. From the properties of the Pauli matrices, it is easy to show that $\hat{O}_{\mathbf{m}}$ has eigenvalues ± 1, regardless of the measurement direction. Since $\mathbf{m}$ is arbitrary, let us we pick three distinct directions, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, such that $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = 0$. The corresponding observables will be $\hat{O}_1 \equiv \mathbf{e}_1 \cdot \boldsymbol{\sigma}$, $\hat{O}_2 \equiv \mathbf{e}_2 \cdot \boldsymbol{\sigma}$, and $\hat{O}_3 \equiv \mathbf{e}_3 \cdot \boldsymbol{\sigma}$. Quantum mechanics not only tells us that measuring $\hat{O}_1$, $\hat{O}_2$, or $\hat{O}_3$ yields ± 1 values, but it also tells us that we *cannot measure them simultaneously,* as they do not commute.

A natural question to ask is the following. Is it possible to assign a value to spin, even though a measurement has not been performed? To answer this, let us assume that we indeed can assign such value (we follow [12, pp. 15–16]). Let $\mathbf{P}$ be a vector random variable corresponding to the actual value of the system's spin before any measurement. It follows that if we measure it in a direction $\mathbf{m}$, the outcome of the experiment must be $\mathbf{m} \cdot \mathbf{P}$. Now, quantum mechanics tells us that, regardless of the direction, $\mathbf{m} \cdot \mathbf{P}$ will take values $+1$ or -1. But, using the vectors we picked before, we have

$$\begin{aligned} \mathbf{e}_1 \cdot \mathbf{P} + \mathbf{e}_2 \cdot \mathbf{P} + \mathbf{e}_3 \cdot \mathbf{P} &= (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \cdot \mathbf{P} \\ &= 0. \end{aligned} \tag{1}$$

This, of course, leads to a contradiction, as the sum of three ± 1 random variables cannot equal zero.

An analysis of the above example shows the origin of the contradiction. Since quantum mechanics forbids the simultaneous measurements of $\hat{O}_k$, as they do not commute, it does not allow us to simultaneously assign values to them. The contradiction does not come from quantum mechanics, but from the assumption that we can assign values to measurements that were not performed. But not even assigning values leads to contradiction, if we are careful. For example, we could assign values to $\mathbf{e}_1 \cdot \mathbf{P}$ and to $\mathbf{e}_2 \cdot \mathbf{P}$, as long as we assumed that the $\mathbf{P}$ in $\mathbf{e}_1 \cdot \mathbf{P}$ is different from the one in $\mathbf{e}_2 \cdot \mathbf{P}$, a feature called contextuality [13].

The above example contains the essence of Nagata's argument. Computing the value of a quantity using the quantum mechanical formalism yields different quantities than computing it from the distribution over the random variables associated with the (non-commuting) observables in quantum mechanics (his use of values from von Neuman's projections). The reason for this discrepancy is that, in the latter case, there is an underlying assumption that an unmeasured quantity exists *independent* of the other quantities. This, of course, is not true, as quantum mechanical variables are contextually dependent from each other, a characteristic stressed by Bohr. We emphasize that this characteristic of quantum mechanics is not at all disturbing, as it is common to many classical systems [14]. The troubling characteristic comes from a combination of contextuality and non-locality, made famous by the Einstein-Podolsky-Rosen paper [15] and by Bell's inequalities [16].

Quantum mechanics is indeed a strange theory. But its strangeness comes not from an inconsistency of its mathematical structure, but from the metaphysical views it imposes on us. If we insist on having worldviews where the values of variables exist independent of the observation, then we will get into contradictions. But, as many authors show, such contradictions can be avoided by carefully interpreting the meaning of the mathematical formalism.

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