

Quantum identity, content, and context: from classical to non-classical logic

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March 4, 2021

Abstract

In this paper, we discuss content and context for quantum properties. We give some examples of why quantum properties are problematic: they depend on the context in a non-trivial way. We then connect this difficulty with properties to the indistinguishability of elementary particles. We argue that one could be in trouble in applying the classical theory of identity to the quantum domain if we take indiscernibility as a core and fundamental concept. Thus, in considering indistinguishability as such a fundamental notion, it implies, if taken earnestly, that one should not apply standard logic to quantum objects. Consequently, we end with a discussion about novel aspects this new mathematics brings and how it relates to some issues associated with the quantum world's ontology and the classical limit. We emphasize that, despite several different ways of questioning classical logic in the quantum domain, our approach is distinct. It involves one of the core concepts of classical logic, namely, identity. So, we are in a different paradigm from standard quantum logics.

1 Introduction

Quantum mechanics (QM) is a very successful theory. It is also a strange theory. Though QM can calculate many experiments' outcomes, there is no consensus about what quantum models tell us about the microscopic world. In other words, it is unclear

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what is the relationship between QM and metaphysics. In this paper, we examine one particular aspect of the quantum world: quantum particles seem to lack identity.

Under certain circumstances, two quantum systems of the same kind (e.g., two electrons) become utterly indistinguishable by any empirical means. However, the lack of identity comes from more than just the impossibility of distinguishing between two quantum particles (e.g., two electrons). It derives from the fact that nothing changes when we permute two identical quantum particles, contrary to what happens in the classical world. This invariance by permutation is at the core of the Bose-Einstein and Fermi-Dirac statistics. In this way, the standard interpretation of the theory assumes indistinguishability. Here, we argue that indistinguishability is an essential concept in quantum theories (both non-relativistic and quantum field theories). Indistinguishability should be thought of as at the same level as celebrated quantum concepts, such as superposition (in particular, entanglement), contextuality, and nonlocality.

Some philosophers and physicists are reluctant to admit that indistinguishability, also known as indiscernibility, plays a salient role in quantum physics's ontology. Perhaps, this reluctance comes from the notion that indistinguishability can be simulated within a "classical" mathematical setting, as we shall see below. However, we find this argument weak for several reasons.

First, just because we can do something does not mean that this is the best approach. Consider, for example, the geometry of curved spaces. We may describe a curved space using Riemannian geometry, where Euclid's postulate of parallel lines is not valid. Alternatively, we can describe the same space by embedding it in a higher-dimensional space and keeping Euclid's postulates. Both approaches yield the same results: all geometry theorems on the curved space are valid in both descriptions. However, one requires a more complicated ontological structure with extra dimensions. Should we make our ontology unnecessarily complicated to accommodate our prejudices? We believe not.

Second, when someone is interested in a theory's foundations, the underlying logic and mathematics become fundamental. We should not do away with an ontological feature because we can use a mathematical trick to describe it. Instead, we argue that the mathematical formalism used to cope with quantum systems' description should consider the ontological features that one aims to describe. Therefore, as we discuss below in more detail, it is crucial to develop a mathematical framework that accommodates indistinguishability in a natural way. In fact, we cannot cope with a contradictory theory (as some claim is Bohr's theory for the atom, yet this is disputable – see the discussion in [39]) within a "classical" framework such as in the mathematics developed in a standard set theory such as the ZFC system, which we presuppose here.¹

Thus, we wish to pursue a metaphysics of non-individuals. In this metaphysics, quantum entities² (here, quantum objects, independently of their proper characterization) are seen as not following the standard notion of identity (to be discussed below).

¹ZFC is the Zermelo-Fraenkel set theory with the Axiom of Choice. The reader can think of it as formalizing the intuitive notion of a set one learned in our math classes.

²The notion of a *quantum object*, or quantum *system*, varies from one approach to another. In orthodox quantum mechanics, we have particles and waves. In the quantum field theories, the basic entities are fields, and particles arise as particular configurations of the fields. Our claims in this paper apply to both particles and fields.

Therefore, we need to change logic and mathematics, unless we accept the physicists' usual way of impersonating them within classical frameworks. These entities need to be considered in most cases as *absolutely* indiscernible, something forbidden in the classical settings.³

Nevertheless, the interpretational problem does not end with the indiscernibility of quantum objects. Indistinguishability is not the only mystery of quantum theory. The ontological status of properties of these objects is also relevant. Quantum properties are tricky, and if we are not careful about how we deal with them, we may reach contradictions. These contradictions arise from considering the possible results of multiple (and incompatible) experiments over the same system. As we have stated elsewhere [9], we *never* perform the same experiment twice. What we do is take a similar experiment, so similar as to be *indistinguishable*. Since experiments are associated with properties, we should consider indiscernible properties also. These indistinguishable properties are also forbidden by classical logic. We need to go outside of standard mathematics and use a different mathematical (and logical) setting as, for example, *quasi-set theory*, to be sketched below. Given that we need to recreate indiscernible properties and systems, it is natural to use a mathematical setting that incorporates indistinguishability as a primitive notion right from the start.

This paper is organized as follows. In Section 2, we first discuss the role of context and content in classical and quantum physics. These two concepts play an essential role in the difficulties physicists and metaphysicists face concerning quantum properties. In Section 3, we consider the concepts of identity and indiscernibility and how they are connected. Identity is a difficult concept, and we explore it both as it is connected to classical physics and indiscernibility in logic. This discussion opens up to our investigations outlined in Section 4. In this section, we argue that by intimately connecting identity to context, we can solve some puzzling aspects of quantum physics. Finally, in Section 6, we outline how to change mathematics to allow for the existence of indiscernibility as a fundamental and primitive concept. This mathematics, grounded on quasi-set theory, captures the idea that quantum objects are indistinguishable and lack a classical identity. As a bonus, we included in Section 7 somewhat more detailed mathematical explanation of the structures discussed in Section 6. We hope the interested reader will find this useful, but this section can be skipped by those readers not seeking further mathematical details. We end the paper with some final remarks, conclusions, and perspectives.

This article is written for a layperson with a strong mathematical background. The reader is assumed to know enough mathematics to be comfortable with logic, set theory, and orthodox quantum mechanics. It should be remarked that a paper dedicated to foundations and aimed at a general reader requires many caveats, since the delicate aspects can be quickly passed unsuspected. We try to warn the reader about those details in between the text or in the footnotes. We ask the reader's forgiveness in advance for the numerous footnotes.

³For a defense of the non-individuals view, see [22].

2 Content and context in quantum and classical physics

The idea of content and context comes from linguistics, specifically semantics and pragmatics. Nevertheless, physics has straightforwardly borrowed those concepts. This section will discuss how content and context translated from linguistics to physics, focusing on quantum mechanics. We organize this section in the following way. First, we concisely review the concepts from linguistics. Then we explore how content and context show up in classical semantics theories. Our discussion should not be thought of as a detailed scholarly review of the linguistic literature on content and context, as this topic is the object of intense research in philosophy of language and linguistics for more than a century. Instead, we present a subset of linguistics that is relevant to physics. With that in mind, we follow our linguistics discussion by examining some physics examples. We see that contents may present context-dependency in both classical and quantum physics. However, we also argue that the context-dependency in quantum physics is different.

Let us start with the concept of content. Roughly speaking, semantic content refers to the meaning of a sentence.⁴ Consider the following statement, made by Vera's friend, Alice:

L1. Vera had a bad date.

Sentence L1 can be seen as a proposition referencing to an object. Assuming the correspondence theory of truth,⁵ its truth value requires some metric, likely subjective, of what constitutes a "bad date." However, once such a metric exists, one could infer L1's truth value. The truth-value of L1, therefore, lies on its semantic content. In other words, a sentence's semantic content can be thought of as a function that takes the sentence and outputs a truth-value.

Context, on the other hand, is the idea that some statements and utterances depend on the circumstances surrounding it, such as time, place, speaker, hearer, and topic, to name a few. For example, Alice's claim that "Vera had a bad date" has different meanings depending on whether their conversation revolved around the fruits of the *Phoenix dactylifera* or romantic engagements. The context alters the meaning and the functions that take the content to truth values.

However, context does not alter meaning only. Consider the case of indexicals. The statement "Acacio is hungry now" is contingent on when it is uttered and on the particular subjective satiety state of the person named "Acacio." In a sense, its meaning does not change. Its referent, Acacio, is the same (assuming we are talking about the same person, one of the co-authors of this paper), the concept of hunger is invariant, and the meaning of now as the present moment is maintained. However, its truth value is

⁴We shall assume this without further discussion, but things are not as straightforward as it may appear. Meaning means "meaning for someone," and there is no meaning *tout court*. Yuri Manin, in his great book [25, pp. 34ff] mentions the case of Lev Alexandrovich Zasedsky, who suffered a brain injury in battle. Zasedsky could write sentences with meaning, such as "An elephant is bigger than an ant," and know that it is true (semantically well defined). But his illness impeded him to understand the meaning of the terms "ant" and "elephant." He had semantics and truth, but not meaning.

⁵We also sustain that the correspondence theory of truth, for instance that treated by Tarski, is not suitable for the empirical sciences, but this is something to be developed in another opportunity; here we take the standard view.

variable. As we write this paragraph, it is false, as Acacio just had lunch. However, the same statement was right about an hour ago. It will be true again several hours from now, even though its meaning is seemingly unchanged.

To summarize, sentences have meanings given by their semantic contents. Sometimes the meanings are context-dependent, as in the case of dates. However, other times, their truth-values vary with context, whereas their meanings seem to do not. We shall see that physics has some correlates to those ideas.

Let us start with classical physics. A physically-relevant proposition about an object is something empirically measurable. For example, we can have the following statement:

P1. A billiard ball's kinetic energy is between $0.1 \text{ kg} \cdot \text{m}^2/\text{s}^2$ and $0.2 \text{ kg} \cdot \text{m}^2/\text{s}^2$.

Similarly to linguistics, P1 has a meaning: if we measure the kinetic energy of a billiard ball, perhaps by measuring its mass and speed and inferring the energy, we find it to be in a certain range. Its meaning is given by an accompanying experimental procedure that yields a truth-value to the sentence. As importantly, this truth-value also corresponds to the idea that the billiard ball, if P1 is true, has a specific property: its kinetic energy.

As in linguistics, P1 refers to a subject (the billiard ball) and a truth-value associated with some meaning-constructing procedure (the experiment). Accordingly, we can think of any physics experiment as observing a physical system's property. This property itself has an associated proposition whose truth-value is assessed by an experiment. So, in a certain sense, properties of physical systems, such as temperature, momentum, energy, present an analogy with contents.

We may take the meaning of a statement as which experiment can yield a truth-value to it. Consequently, expressions such as P1 attach a property to a physical object. Of course, the property is the statement itself, and the experiment is a way to determine its truth-value. To summarize, the properties of a physical system are the content of the propositions.

What about context? Are classical properties context-dependent? Let us examine an example from 18th-century physics. A group of Italian researchers in the 1700s, known as the Experimenters, did not differentiate between heat and temperature but combined both concepts into one (Wiser and Carey, 1983). This combined concept of heat and temperature led to some puzzling results. For instance, the Experimenters wondered about examples such as the following. Imagine we heat a 2 kg piece of iron and immerse it in a container with room temperature water, subsequently measuring the water's temperature. Now, imagine that instead of iron, we use 2 kg of a 3:1 mixture of nitric acid (1.5 kg) and tin (0.5 kg), immersing it in water, as we did with the piece of iron. It was surprising to the Experimenters that even when the mixture of tin and acid was not as hot as the iron, the latter would not raise the water's temperature as much. If both objects, iron and mixture, had the same amount of "hotness," why would they increase the water by different levels of "hotness?"

The answer to the above puzzle is straightforward in contemporary physics, as we distinguish heat and temperature. Because of this distinction, we can measure how much heat a substance holds as their temperature increases: what physicists call specific heat. With this concept, we can measure that iron has a specific heat of 0.44 J/kg K. In contrast, the specific heat of a 3:1 mixture of nitric acid and tin is 1.34 J/kg K. This means that for every one-degree increase in temperature, the amount of heat held by the

2-kg block of iron increases by 0.88 Joules and by 2.64 Joules for the 2-kg tin-nitric acid mixture. In other words, at the same temperature, the mixture holds three times the amount of heat as the iron. Because the Experimenters had a single concept of heat and temperature, they could not even investigate the concept of specific heat, nor could they understand the puzzle.

Let us examine the example above from a slightly different perspective. Imagine we are observing a student who does not distinguish temperature from heat (as the Experimenters) and thinks of both as the smorgasbord concept “hotness.” Consider the following propositions observed to be empirically true for a specific experimental setup involving three objects: X , Y , and W (as for instance X is iron, Y is the mixture of nitric acid and tin, and W is water as in the example above).

A : If X has more heat than Y , then W will have a high temperature.

B : If X has a higher temperature than Y , then W will not have a high temperature.

Both propositions A and B can be true if we carefully chose X and Y 's masses, heat capacities, and how we define statements such as “low temperature,” “high temperature,” and so on. However, let us rephrase A and B in terms of the student's hotness concept. We now have two new propositions, A' and B' :

A' : If X has more hotness than Y , then W will have high hotness.

B' : If X has more hotness than Y , then W will not have high hotness.

A' and B' cannot be both true, as they are contradictory. The contradiction comes here from identifying heat and temperature as a single concept: hotness.

There is an obvious, albeit silly, solution to this contradiction. The student might say, *ad hoc*, that “hotness” in the context of an experiment observing A' is different from experiment B' , so they are not the same statement. To save their hotness concept, the student makes things unnecessarily more complicated than they need to be. As more experiments pile up, the more contexts and the more complicated their theory becomes. Furthermore, such a move would lead to a theory incapable of making good predictions in different situations.

Of course, this is not what scientists usually do. Scientists try to find appropriate ways to describe a physical system that does not lead to contradictions or context dependency. In the hotness case, they realized that differentiating between heat and temperature was consistent and allowed for predictions and explanations of thermal phenomena. When faced with contradictions, scientists realized that the best approach is to face them and figure out ways to rethink our theories or experiments without resorting to context-dependency.

The above example is interesting for historical reasons, but it also illustrates a type of explicit contextuality. In the physics literature, this explicit contextuality is called direct influences [13] or signaling [29]. When the student “explained” the differences between A' and B' as context-dependent, he thought of explicit contextuality. Explicit contextuality manifests when there is a direct contradiction between two statements or results, such as the contradiction between A' and B' . When this happens, scientists

recognize a problem and try to solve it, as with the development of the concepts of heat and temperature.

Let us now move from classical to quantum physics. Quantum physics, as far as we know, forbids any type of properties that exhibit direct influences, i.e., signaling. However, it allow another type of context-dependency (or contextuality): implicit contextuality. In the technical literature, this is called simply “contextuality.” We call it implicit contextuality to emphasize its contrast with contextuality due to direct influences. From now on, when we talk about contextuality, we will refer solely to implicit contextuality.

To understand contextuality in quantum physics, let us consider another example [33]. Imagine a Simon-like-game device with three buttons (instead of the usual 4). Each button on this device, when pushed, randomly emits red or green light. Turns consists of multiple trials, where after observing their behavior, the player can try to predict how each button will lit. For each trial of this game, the player can push at most two buttons at the same time, for as many times as they want, and in any combination of the three buttons they wish. If all three buttons are pushed at the same time, no light is emitted. To win the turn, the player needs to correctly guess what color the unpressed button would light in their last trial.

Let us consider a simple non-contextual example for this game. During her turn, Alice notices the following.

- For trials when she only presses one key, they seem to yield either color randomly. In other words, if Alice presses X , 50% of the time he observes green and 50% red.
- For trials when Alice presses X and Y , she also gets 50% for each color for X or Y , and the two colors are the same;
- For trials when Alice presses X and Z , she also gets 50% for each color for X and Z trials colors are opposite;
- For trials when Alice presses Y and Z , she also gets 50% for each color for Y and Z trials colors are also opposite.

So, after realizing that, if Alice presses X and Y and obtain “red” for both, she could logically infer that Z would be “green.” This is because Z has the opposite color of both X and Y . Guessing “green” would win Alice the turn.

Now, imagine that in another turn, Bob starts prodding different combinations of pairs of X , Y , and Z , and observes the following.

- For trials when Bob only presses one key, they seem to yield either color randomly. In other words, if Bob presses X , 50% of the time he observes green and 50% red.
- For trials when Bob presses X and Y , he also gets 50% for each color for X or Y , but the two colors are the opposite;
- For trials when Bob presses X and Z , he also gets 50% for each color for X and Z trials colors are opposite;

- For trials when Bob presses Y and Z , he also gets 50% for each color for Y and Z trials colors are also opposite.

In other words, when two buttons are pushed simultaneously, they randomly emit red or green light, but in opposite colors. This example exhibits implicit contextuality. To see this contextuality, imagine we start with X emitting green and Y red. Bob can reason that if he pushed X and Z instead, then Z would be red. However, he could also argue that if he pushed Y and Z , since Y was red, Z would be green. Here we reach a logical contradiction: Z would be both red and green, and impossibility in the game. To avoid such contradiction, we need to either assume that Z has no possible color, or that its color changes with the “context” of being seen with X or with Y . To convince themselves that Z changes with which other buttons it is pushed, we urge the readers to think about possible mechanisms that could yield the outcomes we described. The reader will quickly see that any mechanism that generates the outcomes for X and Y needs to be physically different from one generating X and Z (for an example using a firefly in a box, see [8]).

The above example of contextuality is contrived. But contextuality shows up in quantum mechanics. One such example comes from the Greenberger-Horne-Zeilinger state [17], also known as GHZ. Without going into the details of where the following relations are derived, the GHZ state predicts the existence of six observable properties, $X_1, X_2, X_3, Y_1, Y_2,$ and Y_3 , satisfying the following properties. First, the properties X_i and Y_i take values $+1$ or -1 . Second, whenever we observe each of those properties separately, they look completely random, i.e., their average value is zero. The same is true for when we observe them in pairs: they look completely uncorrelated. Third, we can observe them in triples, and when we do, we see the following relationship between the triplets.

$$Y_1 Y_2 Y_3 = 1, \quad (1)$$

$$Y_1 X_2 X_3 = X_1 Y_2 X_3 = X_1 X_2 Y_3 = -1. \quad (2)$$

The above correlations are experimentally observed [7, 4]. Finally, we cannot observe all six properties at the same time. In fact, we can only observe at most three of them simultaneously. For example, quantum mechanics forbids us to see $Y_1, X_1, X_2,$ and X_3 at the same time. Contextuality manifests in a similar way as the previous three-variable example.

To see how contextuality manifests itself, let us assume that the six properties are *not* contextual. Then, we can use (1) and (2) and write the following.

$$(Y_1 X_2 X_3)(X_1 Y_2 X_3)(X_1 X_2 Y_3) = (-1)(-1)(-1) = -1. \quad (3)$$

But we can regroup the above product, and get

$$Y_1 Y_2 Y_3 (X_2 X_3)(X_1 X_3)(X_1 X_2) = Y_1 Y_2 Y_3 (X_2^2)(X_1^2)(X_3^2). \quad (4)$$

However, because X_i is ± 1 valued, their square is 1, i.e., $X_i^2 = 1$. Therefore, it follows that

$$(Y_1 X_2 X_3)(X_1 Y_2 X_3)(X_1 X_2 Y_3) = Y_1 Y_2 Y_3. \quad (5)$$

But this is a mathematical contradiction! The first term in the above equation is -1 whereas the second term is $+1$, and (5) is telling us that $1 = -1$.

Where is the contradiction coming from? It does not come from a mathematical mistake, but from an assumption of non-contextuality. When we wrote that $X_1^1 = 1$, we implicitly assumed that X_1 observed together with Y_2 and X_3 is the same as when observed together with X_2 and Y_3 . This turns out to be false. If we, instead, call each X_i by a different name depending on the context, no contradiction is obtained. What happens in quantum mechanics is similar to the simple color game we discussed before.

The reader may now be thinking about whether we could make a move similar to the contextual classical case. Namely, can we redefine properties such that no such kind of contradictions arise in quantum physics? The answer is yes. Unfortunately, there are many different ways to do so, and there is no consensus among the physics community as to which answer is even acceptable. So, let us end this section with two possible ways around this contradiction.

One move is to assume that properties depend on the context. This is the idea behind Bohm's interpretation of quantum mechanics [3, 19]. In Bohm's theory, the famous duality wave/particle is resolved by assuming both wave and particle existence. The wave fills out the whole of space, and this wave guides the particle. How the wave directs the particle in one direction or another depends on its form. For example, in the two-slit experiment, the wave goes through both slits simultaneously, and due to its interference pattern, it guides the particle toward certain areas and away from others. The result is different if one or two slits are open [19]. Since the wave depends on the context dictated by the physical experiment, Bohm's theory tells us that particles' reality and their properties are contextual. However, Bohm's theory presents a problem: for two or more particles, their waves are affected by their corresponding particle's positions. This theory implies the existence of instantaneous interactions between physical systems. Instantaneous interactions present a difficulty to the causal structure in Bohm's quantum world. As Einstein showed, to have cause and effect, we cannot have instantaneous interactions. This difficulty between Bohm's theory and Einstein's special relativity is the main reason for many physicists to reject it.

Bohm's theory gets into trouble with special relativity because it assumes that properties exist, whether we choose to measure them or not. When we measure, we affect the wave function and, consequently, the physical system. However, the property exists independent of an observer. In other words, Bohm's theory assumes that reality exists, whether we observe it or not.

Another possible solution to the problem of contextuality, particularly to contextuality at a distance (also called non-locality), is to assume quantum properties do not have values before a measurement and that the measurement process "creates" such values. This position was held by Bohr and is the core of the Copenhagen interpretation of quantum mechanics [21]. In this interpretation, saying that an electron has spin $\hbar/2$ in the direction z is meaningless unless we perform a measurement of spin in the direction z and find it to be $\hbar/2$. However, before such a measurement, we cannot say anything about the spin. Furthermore, when we afterward make a measurement of spin in an orthogonal direction, say x , because z and x spins are incompatible (i.e., cannot be measured simultaneously), we cannot say anything anymore about the spin in the z direction; such "property" becomes meaningless. So, Bohr solves the problem of properties in

quantum physics by merely denying their “existence” prior to a measurement.

We shall not cover all possible solutions to defining properties in quantum theory, as they abound. We just wanted to present to the reader two possible paths on how to deal with it and emphasize that the choices we have are not necessarily great. In Bohm’s theory, we need to re-think the concepts of causality and space-time, two well-established tenets of special relativity, to accommodate faster-than-light signaling. In the Copenhagen interpretation, it becomes problematic to talk about a reality independent of a measurement apparatus (and the observer behind it). Either solution present metaphysical difficulties that have troubled physicists for more than a century. These puzzles all boil down to the problem of having properties that depend on the context.

To summarize, in this section, we discussed the idea of content and context. We started with its origins from linguistics and presented an interpretation that allows us to apply these concepts to physical phenomena. We saw that contextual dependencies appear in classical physics, but they are resolved by resorting to reinterpretations and refinements of the theory. We then discussed another contextual dependency that appears in quantum mechanics, such as the GHZ-state example. We then presented some of the proposed solutions to the problems and their corresponding metaphysical issues. In the following sections, we will show that those issues are intimately related to the concept of identity in the quantum world.

3 Identity and indiscernibility

Identity is an old and difficult notion to be dealt with. Usually, the discussions have focused on personal identity and identity through time. Here, we shall be concerned with particular applications of this notion to the identity of objects and properties. By “identity of objects,” or *individuals* as we prefer to call them,⁶ we mean identity of those entities which are dealt with by the theories of physics⁷. For a more detailed discussion about the origins of the term “object,” see [37, pp.13ff]; here we review briefly some aspects of the argumentation given in [14, Chap.1].

We have an intuitive idea of what it means to say that two objects, or individuals, are identical: they are *the same*. However, to say this is to say nothing, for we also do not know what is to be “the same,” something reported equivalent to identity. Thus, we go to the opposite side: we judge individuals as being *different* and, therefore, *not identical*, hence *not the same*. Nevertheless, in virtue of what should individuals be different? Usually, we look for their differences; although quite similar, two peas show differences, maybe some small scratch or a slightly different color. At least, that is what we tend to think.

⁶The word “individual,” according to the Oxford Online Etymological Dictionary, means “one and indivisible.” Hence our preference for the term. However, as it is common practice, we relax the idea of ‘indivisible’ and keep “one,” adding that it can always be distinguished in other contexts, at least in principle, from any other individual as being *that* individual. This distinguishability cannot occur with quantum entities, even those trapped by some device.

⁷The standard quantum formalism is developed within a mathematical structure called “Hilbert-space formalism,” although there are alternatives ([34] mentions *nine* different ways of developing orthodox quantum mechanics).

Still, in virtue of what two objects would be different? Are they so? Is it possible to have two (or more) objects perfectly alike, with no differences at all? Put in other words, what makes an object an individual, distinct from any other? Is there some Principle of Individuation we can use to specify an individual's individuality? Theories of *individuation* are generally divided up in two main lines: *substratum theories* and *theories of bundles of properties*. According to the first group, beyond the properties of an object and the relations it can share with others, there is *something more*, something Locke described as "I don't know what" [24, Book I, XXIII, 2]. This notion and the related ones (such as haecceities and thisness)⁸ were discarded in favor of bundle theories of individuation. Bundle theories say that there is nothing more to an object than the collection of its properties (encompassing relations). Nevertheless, if in the substratum theories one could say that what distinguishes an object from another is its substratum (or something like that), in bundle theories, many discussions have appeared concerning the possibility of two objects having the same collection of properties. Can they have the same collection of properties? If not, why not? Of course, that objects in our scale, i.e., "macroscopic objects," can partake all their properties is something that cannot be logically proven. This assumption must be accepted as a metaphysical hypothesis, and there are no known counterexamples to it. Furthermore, this hypothesis was what Western philosophy has preferred, from the Stoics to Leibniz's metaphysics.

Let us remember Leibniz's metaphysics' intuitive idea: *no two individuals share all their properties; if they have the same attributes, they are not different, but the same individual*. This metaphysical principle was encapsulated in standard logic with the definition of identity given by Leibniz Law. This law says what we have expected: entities are identical if and only if they share all their properties, hence all their relations, that is, if and only if they are indistinguishable.

What about the identity of properties? In standard logic, we usually say that two properties, P and Q , are "identical" if they are satisfied by the same "things." For instance, for Aristotle, the properties "to be a human" and "to be a rational animal" are "identical" in this sense. As an example from standard mathematics, consider the sets $\{x \in \mathbb{R} | x^2 - 5x + 6 = 0\}$, $\{x \in \mathbb{N} | 1 < x < 4\}$, and $\{x \in \mathbb{R} | x = 2 \vee x = 3\}$. These three sets are identical: they have the same extensions but different *intensions*.⁹

Classical mathematical frameworks do not accommodate indistinguishables; entities sharing all their attributes and being just numerically distinct do not exist in classical mathematics (but see below). Individuals are unique, separable, at least in principle, counted as one of a kind and presenting differences to every *other* object. There are no purely numerical identical individuals: some form of Leibniz's Law holds. This is so within standard logic and mathematics, and the ways of dealing with indiscernibles require mathematical tricks such as confining them to non-rigid structures¹⁰. For

⁸There are peculiarities in using these terms, but broadly speaking, all refer to something beyond an individual's properties.

⁹In technical terms, in extensional higher-order logics, we can define such a notion by saying that P and Q are identical when they have the same *extensions*, that is, when they are satisfied by the same lower terms.

¹⁰A structure (a domain comprising relations over its elements) is rigid if its only automorphism (bijections that preserve the relations of the structure) is the identity function. Indiscernibility in a structure means that the objects are invariant by some automorphism of the structure; in rigid structures, an object is indiscernible just from itself. Non-rigid (deformable) structures hide the object's identity so that we may not be able to discern them by lack of distinctive relations or properties. For details, see [14, §6.5.2], [23].

example, take the structure $\langle \mathbb{Z}, + \rangle$, which represents the integer numbers, \mathbb{Z} , and *only* the standard addition operation, “+.” This structure is not rigid, since the transformation $f(x) = -x$ is an automorphism of the structure, i.e., it keeps the individuals indiscernible within its point of view. To see this, take the 2 and -2 . We cannot discern them *within* this structure. Imagine any property for 2 defined only with “+,” such as “ $2 + 1 = 3$.” If we change the numbers by the “minus” ones, we have “ $(-2) + (-1) = (-3)$.” From within this structure, the latter is identical to the former; we cannot distinguish them. Of course, if we added additional properties to the structure, such as the “ $<$ ” relation, it would become rigid, and we would be able to distinguish between 2 and -2 . However, we cannot do it only with “+.”

The search for *legitimate* indiscernible objects/individuals, in the above sense and without mathematical tricks, requires a change of logic. We will retake this discussion later on this paper, but we wish to turn to another kind of question for now.

Some authors, such as Peter Geach, argue that identity is relative. The only thing we can say, according to him, is that two individuals a and b are (or not) identical relative to a sortal¹¹ predicate F ; in the positive case, we say that they are F -identical and can write $a =_F b$. In our opinion, identity is absolute. Identity is, according to us, to be associated with metaphysical identity, as explained above. It is something an individual has that says that it is unique and, when it appears in some other context, we are authorized to think that it is the same individual that has appeared twice. Alternatively, an individual’s identity is its identity card, one for each individual: it accompanies it in all contexts and, with its help, we can distinguish the individual as being *the same* individual of a previous experience. Identity makes the individual’s name a rigid designator, denoting the same entity in all possible accessible worlds. As it is well known, David Hume guessed that there is no such an identity; according to him, we recognize someone as being *the same* from a previous experience by habit, by familiarity [20, p.74 and *passim*], but cannot “logically” prove that. Schrödinger had a similar opinion regarding quantum entities when he says that

“[w]hen a familiar object reenters our ken, it is usually recognized as a continuation of previous appearances, as being the same thing. The relative permanence of individual pieces of matter is the most momentous feature of both everyday life and scientific experience. If a familiar article, say an earthenware jug, disappears from your room, you are quite sure that somebody must have taken it away. If after a time it reappears, you may doubt whether it really is the same one – breakable objects in such circumstances are often not. You may not be able to decide the issue, but you will have no doubt that the doubtful sameness has an indisputable meaning – that there is an unambiguous answer to your query. So firm is our belief in the continuity of the unobserved parts of the string!” [32, p.204]

Entities partaking metaphysical identity are termed *individuals*. Can we think of *non-individuals* too? If yes, can we give examples of entities of this kind? The first way to

¹¹A sortal predicate enables to count the objects that obey the predicate, such as “being a philosopher.” So, Isaac Newton and Stephen Hacking would both be counted as “Lucasian Professor of Mathematics in Cambridge.”

think of them, by considering what we have said, is to deny them the epithet “to have an identity.” What should it mean? The short answer is that they would be entities sharing all their characteristics, either substratum or properties and relations. From now on, we shall avoid speaking of substratum and keep with bundle theories [36]. However, non-individuals, in our formulation, are not simply metaphysically or numerically identical entities, although this is logically possible.¹² Our notion is weaker, enabling non-individuals to form collections (termed “quasi-sets”) with cardinalities greater than one so that no particular differences can be ascribed to them. Furthermore, they would be indistinguishable even if an omniscient demon (Laplace’s demon) exchanged them with one another; in this case, nothing would change in the world at all. That is the difference: individuals, by definition, when permuted, make a difference! This difference is of fundamental importance, for it involves several other related notions which appear in physical theories, such as space and time and, fundamentally, permutations. We shall need to explain that further, but for now, we wish to emphasize that we do not regard identity as something an entity *must* have. When something has an identity, then it is absolute, it is metaphysical, and no two entities with identity can be only numerically distinct. *Non-individuals* are entities that lack identity, that can be just numerically discerned, that have all the same identity card. If one looks at one non-individual here and there, one finds “another” one in a different context; not even demons or gods will tell one if this new object is “different” or “the same” one found previously, as this would be meaningless.

Nevertheless, once we think about more than one entity, one could claim that they must be *different*. Mathematically, this would be expressed by the set-theoretical argument that once the cardinal of a set is greater than one, its elements *must* be different. We stress that this depends on the set theory one is taking into account. In standard set theories, such as the most celebrated systems (the apparently most famous one is termed “ZFC”), this is true, but in *quasi-set theory* (discussed below), this is may not be the case. In quasi-set theory, we not only can have collections (quasi-sets) of absolutely indiscernible entities and with a cardinal greater than one, but we can also quantify such “non-individuals.” Quasi-set theory shows that Quine’s motto of “no entity without identity” [30, p.23] does not hold in general, for even non-individuals can be values of the variables of a regimented language.

3.1 Identity in classical formal settings

There is a problem concerning the metaphysical identity of the last section: it cannot be defined in first-order languages [18, 14].¹³ We provide here a slightly technical explanation. As said earlier, first-order languages speak of the individuals of some

¹²In his criticism to the definition of identity given by Whitehead and Russell in their *Principia Mathematica* (Leibniz Law, in a standard second-order language, $x = y := \forall F(Fx \leftrightarrow Fy)$, where x and y are individual terms and F is a predicate variable for individuals), F. P. Ramsey said precisely this: that we could logically conceive entities violating the definition, sharing all their properties, and even so not being the same entity [31, p.30].

¹³First-order languages deal with domains of individuals, their properties, relations and operations over them. Quantified expressions like “There exists some x such that . . .” and “For all individuals x , . . .” applies only to individuals, and we cannot say things like “There is a relation among individuals . . .” or “For every property of individuals . . .” In logic, we say that first-order languages quantify over individuals only.

domain. Usually, the axiomatizations take logical identity as primitive (represented by a binary predicate “=”), subject to certain axioms (reflexivity and substitutivity). We can prove that identity is an equivalence relation, really a congruence, whose intended interpretation is the *identity of the domain*; calling it D , then we are referring to the set $\Delta_D := \{\langle a, a \rangle : a \in D\}$, also called the *diagonal of D* . But it can be proven that there are other structures, called elementary equivalent structures,¹⁴ which also model “=” but interprets this symbol in sets other than the diagonal (op.cit.). So, within a first-order language, we never know if we speak of the identity (or the difference) of two individuals or of, say, classes of individuals.

Higher-order languages enable us to define logical identity by Leibniz Law, but such logical identity is defined through indiscernibility. If we wish to define indiscernibility instead, the definition would be the same: agreement for all properties. So, higher-order languages do not distinguish between these two concepts. If we intend to speak of indiscernible but not identical things, Leibniz Law does not help.¹⁵ Furthermore, if we aim to preserve some meta-properties of our system (Henkin’s completeness), we are subject to find Henkin models so that two objects of the domain look as indiscernible since they obey all the language’s predicates, but which are not the same element [14, §6.3.2]. In short, we need to conclude that metaphysical identity cannot be defined. The most we can do is find refuge in logical identity, but this, as we shall see soon, causes troubles to quantum mechanics.

However, let us first put away the often-made claim that even quantum objects can be discerned by spatio-temporal location.

3.2 Identity and space and time

There is still another way to look at identity in classical settings: include space and time. Orthodox non-relativistic quantum mechanics makes use of *classical* space and time or, as we can say, “Newtonian” absolute notions. Intuitively, the classical space and time structure is a space that looks, at least for small regions, like the \mathbb{R}^4 , namely three dimensions for space (\mathbb{R}^3) and one for time (\mathbb{R}). More precisely, mathematically, the classical space-time is a manifold locally isomorphic to \mathbb{R}^4 , usually termed \mathbb{E}^4 (for “Euclidean”); see [28, Chap. 17].

This structure has some interesting features, but for us here, an important characteristic is that it is a “Hausdorff space.” This property of being Hausdorff means that, given any two points a and b , $a \neq b$, it is always possible to find two disjoint open sets (say two open balls) B_a and B_b such that $a \in B_a$ and $b \in B_b$. In extensional contexts, such as the ZFC set theory, a property is confounded with a set; the objects that belong to the set are precisely those satisfying the property. So, a and b have each a property not shared with the other, namely, to belong to “its” open set. Hence, Leibniz’s Law applies, and they are different. Notice that this holds for any *two* objects a and b : once we have *two*, they are distinct. Therefore, we may say that, within such a framework,

¹⁴Elementary equivalent structures are interpretations of a first-order language that preserve the same truth sentences. From the language’s point of view, one cannot distinguish among such structures: they look the same.

¹⁵The distinction between identity and indiscernibility can be made only in semantical terms; see [6].

there are no indiscernibles!¹⁶

Let us see now how we can pretend to say that we have indiscernibles within a classical framework.

3.3 Indiscernibility in classical logical settings

Still working in a classical setting, say the ZFC system, we can mimic indiscernibility. In this subsection we expand the above discussion about using non-rigid structures, presenting some of its more technical concepts and ideas.

Usually, we say that the elements of a certain equivalence class are indiscernible, and perhaps this is acceptable for certain purposes. More technically, in doing that, we are restricted to a *non-rigid (or deformable) structure*. As we saw previously, we say that a structure $\mathfrak{A} = \langle D, R_i \rangle, i \in I$, is *rigid* if its only automorphism is the identity function; this means that we have a domain D , a non-empty set, and a collection of relations over the elements of D , each one of a certain arity $n = 0, 1, 2, 3, \dots$ ¹⁷ If the structure is not rigid, then it is non-rigid or deformable. We saw an example of a deformable structure earlier on, the $\langle \mathbf{Z}, + \rangle$. Another example of a deformable structure is the field of the complex numbers, for the operation of taking the conjugate is an automorphism. In such a structure $\mathfrak{C} = \langle \mathbb{C}, 0, 1, +, \cdot \rangle$, the individuals i and $-i$ are indiscernible.

Given \mathfrak{A} as above, we say that the elements a and b of D are \mathfrak{A} -indiscernible if there exists $X \subseteq D$ such that (i) for every automorphism h of \mathfrak{A} , $h(X) = X$, that is, X is invariant by the automorphisms of the structure, and (ii) $a \in X$ iff $b \notin X$. Otherwise, a and b are \mathfrak{A} -discernible [23].

It is clear that in a rigid structure, the only element indistinguishable from a is a itself since the only automorphism is the identity function. In informal parlance, we may say that a and b are \mathfrak{A} -indiscernible iff they are invariant by permutations that “preserve the relations of the structure.”

Something like that is what we do in quantum mechanics. Roughly speaking, the theory says that when we measure a certain observable value for a quantum system in a certain state, the value does not change before and after a permutation of particles of the same kind. Physicists say that *permutations are not observable*, and this is expressed by the Indistinguishability Postulate.¹⁸

¹⁶In model theory, an important part of logic, we can speak of “indiscernibles” in a sense, for instance, *Ramsey indiscernibles*. However, this is a way of speaking; even these entities obey the classical theory of identity, therefore being individuals. See [5, Chap.15].

¹⁷That the identity mapping is an automorphism is trivial. For all the argumentation, it is enough to consider *relational structures*, for distinguished elements and operational symbols can be taken as particular kinds of relations; also, we subsume all domains in just one.

¹⁸In technical terms, let us take a permutation P between particles denoted by x_i and x_j . As usually stated, we may say that for any x_1, \dots, x_n ,

$$P(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \leftrightarrow P(x_1, \dots, x_j, \dots, x_i, \dots, x_n) \quad (6)$$

The *Indistinguishability Postulate* is expressed in terms of “expectation values;” it says that

$$\langle \psi | \hat{A} | \psi \rangle = \langle P\psi | \hat{A} | P\psi \rangle \quad (7)$$

for any observable represented by a self-adjoint operator \hat{A} and for any permutation operator P , being $|\psi\rangle$ the vector state of the system.

Leaving formal logic and mathematics for a while, let us consider more general situations, which will lead us to a more detailed discussion about quantum mechanics. We shall commence by emphasizing the importance of the *contexts*.

4 Connecting identity to context

On many occasions, we are tempted to think about possible worlds which are not actual. We wonder what our life would have been like if we had taken different decisions at crucial moments. We can think about an object, person, or animal, in many different circumstances, which can differ from the actual ones. For example, suppose that we have a pet cat and live in a small apartment. Given its living conditions, the cat cannot catch the birds that he sees through the window. He observes them with attention, craving for them but unable to reach them. Thus, in our tiny-apartment world, our cat never caught a bird. Furthermore, he never will because he cannot go out. However, we can *imagine* a different world, in which we live in a house with a big yard in which our cat can wander out as many times as it wants. In this big yard world, our cat can surely try to catch a bird, and he will undoubtedly do so at least once.

The above story is an example of how we reason about counterfactuals. We are tempted to conclude something that occurs in a world that is not actual *could* happen, even if that world never becomes actual. This kind of reasoning is very natural in our everyday life. However, what are the assumptions behind it? First, somehow, our cat retains its identity among the different worlds: the cat in the small apartment world is the same as the cat in the big yard world. Both cats have the same name, color, same capabilities, and desire to catch birds. Nevertheless, how can we assure that the cat will retain its properties among the different worlds? Perhaps, if we could afford a house with a big yard, we could also afford fancy and tasty cat food. The cat gets used to it, stays inside the house, and eats the whole day. In the fancy house world, it might become idle to the point that it barely moves or plays, as it happens with some cats. When it finally goes out to the garden, it cannot catch birds anymore, as it became clumsy and slow.

The above example shows that we should not make hasty conclusions: the properties of an object, person, or animal, might depend strongly on the *context* in which we are considering them. In the small apartment, humble life, with cheap food, our cat is playful and agile: it has a high probability of catching a bird but no bird to catch. In the big house, those properties may or may not be valid. The first lesson is: to assume that an object retains its properties among different and incompatible worlds is not granted. Even more so, one may ask: in which sense are the two cats in different worlds the same? From a strict point of view, one may say that the agile cat from our actual world is not the same as the idle cat of the alternative reality. In the same way, we should not mix the different worlds with counterfactual reasoning. If we conclude, by studying our cat in this actual world, that he is very skilled in chasing birds, we cannot use empirical information from our world to conclude that the cat will indeed chase a bird in the alternative world.

Thus, we are introduced to a profound philosophical problem by thinking about the above straightforward situation: what are the principles or conditions that grant identity

to objects considered in different possible worlds? Are we entitled to say that a given object retains its identity when considered in different and incompatible situations? Of course, in many situations of our daily life, assuming that objects retain their identities and properties in different contexts will work. Our bike works well on sunny and rainy days and in diverse landscapes (such as cities or mountains). Many characteristics of our bike – such as its color or its range of velocities – are, to a great extent, *context independent*. However, we should not take this context independence for granted. This is more so if we consider quantum systems that define phenomena that lie far beyond our everyday experience. The realm of the atom extends far beyond the ångström scale (ten to the minus ten meters, which is something like 0,0000000001 meters for one ångström!). The principles – whatever they are – that allow us to identify properties and objects among incompatible situations may no longer be valid for atomic systems. Moreover, this seems to be the case, as the GHZ example above and the following example show.

Suppose that Alice and Bob have separated labs, L_A and L_B , in which they perform their experiments. At a given time, a third party prepares a quantum system capable of affecting what happens in L_A and L_B . Suppose that Alice decides to make an experiment P_A in her lab, in order to interact with the given quantum system, and that Bob can do P_B or P'_B in L_B . Due to the peculiarities of quantum mechanics, P_B and P'_B cannot be performed at the same time – they are *incompatible* experiments. To understand what *incompatible* means, imagine the following situation: in order to perform P_B , Bob must align a magnet in a given direction d , and in order to perform P'_B , he must align its magnet in a different direction d' . A magnet cannot point in two different directions – similarly, a clock's handle cannot point at two different angles simultaneously. Thus, there are two incompatible situations: either Alice performs experiment P_A and Bob performs P_B , or Alice performs P_A and Bob P'_B . The two possibilities *cannot* coexist in the same world. Let us call these possibilities W_1 and W_2 , respectively.

Suppose now that Alice and Bob are in the process of deciding what to do. They wonder about the experiments' possible outcomes in the different situations, W_1 and W_2 . Notice that they do not need actually to perform the experiments. It is all about reasoning in various alternatives without actually performing them. Now we question: what is the status of the possible results of experiment P_A concerning W_1 and W_2 ? After the discussion about the cat, we should not be as quick to identify what happens in W_1 with W_2 , even if we are talking about the same experiment, P_A . In both possible worlds, Alice will perform the same actions (she will orient the magnets in the same directions, prepare the same reading apparatus, and so on). Is she going to obtain the same results? What enables us to conclude that she will? Notice that we are not asking here about an *influence* of Bob's actions in Alice ones: the laboratories can be very far away in space and time. We are asking here whether we are entitled to assume that there is some trace of identity among the results obtained in different (and incompatible *worlds*). As expected, the answer is: no, we are not. Contradictions can be readily achieved if we do so, as the cat and contextuality examples suggest (and shown in technical research on quantum theory).

The actions required for experimenting P_A are the same in W_1 and W_2 . Can we say that P_A in W_1 is the same as P_A in W_2 ? After the cat discussion, let us be conservative about the answer. We will say that P_A in W_1 is *indistinguishable* from P_A in W_2 . The

two experiments are completely alike: Alice will execute the same actions in a system prepared with an equivalent procedure in both worlds. However, we should not be tempted to claim they are the same. The more so, we should not expect the same results. In this sense, we say that the properties studied by experiment P_A in W_1 are indistinguishable from the properties studied by P_A in W_2 . We denote these properties by the pairs $(P_A; W_1)$ and $(P_A; W_2)$ and write $(P_A; W_1) \equiv (P_A; W_2)$, to stress the fact that they are indistinguishable (but not identical). A natural, logical formalism for describing this kind of indistinguishability is the quasi-set theory. This theory allows us to consider properties or objects in alternative worlds as collections of indiscernible ur-elements.

If world W_1 becomes actual, Alice and Bob will perform their actions, obtain their results, and record them. Out of these results, what conclusions should they take about the possible results associated with W_2 ? Are they entitled to reason in a counterfactual way and combine the results of worlds W_1 and W_2 to extract conclusions about them? Much caution should be taken here, as the cat and contextual examples show. In principle, there is no *a priori* reason to do so. That we are allowed to do so in many (but not all!) everyday situations is more a lucky strike that we share with other creatures in our macroscopic reality than a general rule. Counterfactual reasoning simplifies our existence, but we should not expect it to be valid in every situation. This lack of validity seems empirically suggested at microscopic scales, which are very different from our own.

To summarize, we can state the following:

- Even if state preparations and measurement procedures are completely alike among different worlds, we should not treat them as identical. In this sense, we speak about things such as indistinguishable properties and objects.
- Even if two experiments are completely indistinguishable, we should not expect the same results in different worlds.
- We should not derive conclusions from counterfactual reasoning, especially in the quantum domain. Such conclusions are not reliable and are not metaphysically justified.

5 Quantum mechanics in classical logical settings

In this section, we briefly review how the standard quantum formalism performs the trick of treating indiscernible quantum systems within the scope of classical logic (encompassing mathematics). In doing so, we lay the groundwork for alternative logics and mathematics, which provide an adequate description from our perspective.

A glance at standard textbooks on quantum mechanics reveals that they use classical mathematics, hence classical logic. However, the claim that quantum mechanics requires a different logic, known as quantum logic, can also often be found.¹⁹ These two observations seem contradictory. Why is this apparent contradiction present in the literature?

¹⁹The field of “quantum logic” arose from Birkoff and von Neumann’s 1936 seminal paper. The reader interested in the subject is referred to the following excellent papers: [12] and [35].

The reason may be as follows. Most physicists are concerned with physical problems being solved by quantum theory and not with philosophical or logical foundational questions about it. Although they might endorse some particular interpretation of quantum mechanics, thus presupposing some concern with quantum theory's philosophy, most physicists use "classical" mathematics in an almost instrumentalist way. Thus, when dealing with entities that would be indistinguishable, physicists use some mathematical tricks to hide the identifications typical of our standard mathematical languages. Let us see how they do it.

First, we recall that, in quantum mechanics' standard formulation, a system's state is represented mathematically by a vector in a Hilbert space. This vector, also called the wave function, is supposed to encode all information available for that system in a specific situation. Observables, which represent possible experimental procedures and their outcomes, are self-adjoint operators in the Hilbert space. When an observable is measured, the state-vector enters (or "collapse") into one of the observable operator's eigenvectors. Since this process is "mysterious," in the sense that the formalism does not explain how it happens, many physicists try to avoid it, adopting alternative explanations. Nevertheless, the primary mathematical object in quantum theory is the Hilbert space and vectors in it. So, the question is how to represent indistinguishable objects using the mathematics of vectors.

Quantum particles come in two types: bosons and fermions. Their main difference comes from their statistics: bosons follow the Bose-Einstein statistics, whereas Fermions satisfy the Fermi-Dirac one. Both statistics count objects as if they were indistinguishable, contrary to the classical Maxwell-Boltzman statistics.

Bosons are a typical type of indistinguishable quantum entities. Bosons are a kind of quantum "particles," and they are entirely indistinguishable when prepared in the same quantum state. This state is such that they share all the relevant quantum properties. A system composed of, say, two bosons 1 and 2 in two possible situations *A* and *B* is described by a symmetric wave function such as the following.

$$\Psi = \frac{1}{\sqrt{2}}(\psi_1^A\psi_2^B + \psi_2^A\psi_1^B), \quad (8)$$

where $\psi_1^A\psi_2^B$ means system 1 in the state *A* and system 2 in *B* and similarly for the other term. The $\frac{1}{\sqrt{2}}$ is just a normalization factor required by the formalism. Ψ is invariant under the permutation of 1 and 2. This invariance means that exchanging particle 1 by 2 (and vice-versa) does not affect the state of the system. Consequently, any measurement results are maintained under permutations.

This symmetrization of the wave function works, but it is a trick. We are still using labels to "name" the particles because our language and mental models have a hard time thinking otherwise. In other words, this trick assumes, upfront, that bosons are individuals. Suddenly, as if a miracle happened, permutations do not conduce to different situations. However, this invariance was put there by hand. We could give more detailed arguments as to why this is a mathematical trick that does not make bosons indistinguishable, but we hope the above example is sufficient for the reader to grasp the main idea.

The use of the above trick is similar to confining the discussion to a deformable (non-rigid) structure, as explained earlier. However, as mentioned, within such classical

settings, we can always go “outside” of the structure and identify the particles. This possibility of identification is at odds with the hypothesis that they are indiscernible.²⁰

There is no way to escape this conclusion. As we have said before, standard mathematics and logic are theories of individuals. This is so for historical reasons: classical logic, mathematics, and even classical physics were built with individuals in mind. Quantum mechanics, of course, came to challenge those ideas and to question the concepts of individuality.

6 Alternative logical approaches

Assuming that indiscernibility is a core notion in quantum mechanics, we should look for an alternative logical and mathematical basis that considers it right from the start. This bottom-up approach would not mimic it within a standard framework from a top-bottom one. Our strategy is grounded in a metaphysics of non-individuals (for detail, see [14], [22], and references therein). Moreover, it tries to develop mathematics compatible with such metaphysics. Consequently, Schrödinger logics and quasi-set theory were developed in the 1990s. Although they are mathematical developments independent of the interpretations, the intended one is precisely to cope with such non-individual entities. In this section, we will give a rough idea about how quasi-set theory works. For a review about Schrödinger logics, see [14, chap.8].

6.1 Quasi-set theory

In the quasi-theory \mathfrak{Q} , indiscernibility is a primitive concept, formalized by a binary relation “ \equiv ” satisfying the properties of an equivalence relation, but not full substitutivity.²¹ In this notation, “ $x \equiv y$ ” is thought to mean “ x is indiscernible from y .” This binary relation is a partial congruence in the following sense: for most relations, if $R(x, y)$ and $x \equiv x'$, then $R(x', y)$ as well (the same holds for the second variable). The only relation to which this result does not hold is membership: $x \in y$ and $x' \equiv x$ does not entail that $x' \in y$; details in [14, 15]).

Quasi-sets can have as elements other quasi-sets, particular quasi-sets termed *sets* which are copies of the sets in a standard theory (in the case, the Zermelo-Fraenkel set theory with the Axiom of Choice), and two kinds of atoms (entities which are not sets), termed *M*-atoms (*M*-objects), which are copies of a standard set theory with atoms (ZFA) and *m*-atoms (*m*-objects), which have the quanta as their intended interpretation, to whom it is supposed that the logical identity does not apply. If we eliminate the *m*-atoms, we are left with a copy of ZFA, the Zermelo-Fraenkel set theory with atoms. Hence, we can reconstruct all standard mathematics within \mathfrak{Q} in such a “classical part” of the theory.

Functions cannot be defined in the standard way. When *m*-atoms are present, it cannot distinguish between indiscernible arguments or values. Therefore, the theory

²⁰The way to “go outside” the quantum formalism is to go to the set-theoretical universe since all mathematics used in quantum mechanics can be performed in terms of sets.

²¹If we add substitutivity to the postulates, then no differences between indiscernibility and logical first-order identity would be made.

generalizes the concept to “quasi-functions,” which map indiscernible elements into indiscernible elements. See below for more on this point.

Cardinals (termed “quasi-cardinals,” qc) are also taken as primitive, although they can be proven to exist for finite qsets (finite in the usual sense [10, 2]). The concept of quasi-cardinals can be used to speak of “several objects.” So, when we say that we have two indiscernible q -functions, according to the above definition, we are saying that we have a qset whose elements are indiscernible q -functions and whose q -cardinal is two.²² The same happens in other situations.

An interesting fact is that qsets composed of several indistinguishable m -atoms do not have an associated ordinal. This lack of an ordinal means that these elements cannot be counted since they cannot be ordered. However, we can still speak of a collection’s cardinal, termed its *quasi-cardinal* or just its *q-cardinal*. This existence of a cardinal but not of an ordinal is similar to what we have in QM when we say that we have some quantity of systems of the same kind but cannot individuate or count them, e.g., the six electrons in the level $2p$ of a Sodium atom.²³

Identity (termed *extensional identity*) “ $=_E$ ” is defined for qsets having the same elements (in the sense that if an element belongs to one of them, then it belongs to the another)²⁴ or for M -objects belonging to the same qsets. It can be proven that this identity has all the properties of classical logical identity for the objects to which it applies. However, it does not make sense for q -objects. That is, $x =_E y$ does not have any meaning in the theory if x and y are m -objects. It is similar to speak of categories in the Zermelo-Fraenkel set theory (supposed consistent). The theory cannot capture the concept, yet it can be expressed in its language. From now on, we shall abbreviate “ $=_E$ ” by “ $=$,” as usual.

The postulates of \mathfrak{Q} are similar to those of ZFU, but by considering that now we may have m -objects. The notion of indistinguishability is extended to qsets through an axiom that says that two qsets with the same q -cardinal and having the same “quantity” (we use q -cardinals to express this) of elements of the same kind (indistinguishable among them) are indiscernible too. As an example, consider the following: two sulfuric acid molecules H_2SO_4 are seen as indistinguishable qsets, for both contain q -cardinal equals to 7 (counting the atoms as basic elements), and the elements of the sub-collections of elements of the same kind are also of the same q -cardinal (2, 1, and 4 respectively). Then we can state that “ $H_2SO_4 \equiv H_2SO_4$,” but of course, we cannot say that “ $H_2SO_4 =$

²²Quasi-cardinals turn to be *sets*, so we can use the equality symbol among them. We use the notation $qc(x) = n$ (really, $qc(x) =_E n$, see below) for a quasi-set x whose cardinal is n .

²³To count a finite number of elements, say 4, is to define a bijection from the set with these elements to the ordinal $4 = \{0, 1, 2, 3\}$. This counting requires that we identify the elements of the first set.

²⁴There are subtleties that require us to provide further explanations. In \mathfrak{Q} , you cannot do the maths and decide either a certain m -object belongs or not to a qset; this requires identity, as you need to identify the object you are referring to.

In quasi-set theory, however, one can hypothesize that *if* a specific object belongs to a qset, then so and so. This is similar to Russell’s use of the axioms of infinite (I) and choice (C) in his theory of types, which assume the existence of certain classes that cannot be constructed, so going against Russell’s constructibility thesis. What was Russell’s answer? He transformed all sentences α whose proofs depend on these axioms into conditionals of the form $I \rightarrow \alpha$ and $C \rightarrow \alpha$. Hence, *if* the axioms hold, *then* we can get α . We are applying the same reasoning here: *if* the objects of a qset belong to the another and vice-versa, *then* they are extensionally identical. It should be noted that the definition of extensional identity holds only for sets and M -objects.

H₂SO₄,” as for in the latter, the two molecules would not be two at all, but just the same molecule (supposing, of course, that “=” stands for classical logical identity). In the first case, notwithstanding, they count as two, yet we cannot say which is which.

Let us speak a little bit more about quasi-functions. Since physicists and mathematicians may want to talk about random variables over qsets as a way to model physical processes, it is important to define functions between qsets. This can be done straightforwardly, and here we consider binary relations and unary functions only. Such definitions can easily be extended to more complicated multi-valued functions. A (binary) q-relation between the qsets A and B is a qset of pairs of elements (sub-collections with q-cardinal equals 2), one in A , the other in B .²⁵ Quasi-functions (q-functions) from A to B are binary relations between A and B such that if the pairs (qsets) with a and b and with a' and b' belong to it and if $a \equiv a'$, then $b \equiv b'$ (with a 's belonging to A and the b 's to B). In other words, a q-function maps indistinguishable elements into indistinguishable elements. When there are no m -objects involved, the indistinguishability relation collapses in the extensional identity, and the definition turns to be equivalent to the classical one. In particular, a q-function from a “classical” set such as $\{1, -1\}$ to a qset of indiscernible q-objects with q-cardinal 2 can be defined so that we cannot know which q-object is associated with each number (this example will be used below).

To summarize, in this section, we showed that the concept of indistinguishability, which conflicts with Leibniz’s Principle of the Identity of Indiscernibles, can be incorporated as a metaphysical principle in a modified set theory with indistinguishable elements. This theory contains “copies” of the Zermelo-Frankel axioms with *Urelemente* as a particular case when no indistinguishable q -objects are involved. This theory will provide us the mathematical basis for formally talking about indistinguishable properties, which we will show can be used in a theory of quantum properties. We will see in the next section how we can use those indistinguishable properties to avoid contradictions in quantum contextual settings such as KS.

7 Formulating quantum mechanics within quasi-set theory

As we have seen, the quasi-set theory enables us to form collections (the quasi-sets) of “absolutely” indiscernible elements. In this theory, even if one goes outside the relevant structures, they will not become rigid: this mathematical universe is not rigid. Thus, the quasi-set theory is a suitable device to develop a quantum theory where indiscernibility is considered from the start as a fundamental notion. This section explains how quantum mechanics (in the Fock space formalism) can be developed within the quasi-set theory \mathcal{Q} . The current development is based in [11] and is technical. This level of mathematical formality is necessary to provide essential details. The reader unconcerned with such technicalities may skip this section and proceed directly to the conclusions.

²⁵We are avoiding the long and boring definitions, as, for instance, the definition of ordered pairs, which presuppose lots of preliminary concepts, just to focus on the basic ideas. For details, the interested reader can see the indicated references.

7.1 The \mathfrak{Q} -spaces

In the standard mathematical formalisms, the assumptions that quantum entities of the same kind must be indiscernible are hidden behind mathematical tricks such as symmetrizing wave-functions and vectors. In order to avoid these tricks, we introduce the notion of \mathfrak{Q} -spaces. The resulting framework is termed *nonreflexive quantum mechanics or, simply, nonreflexive*.

We begin with a q-set of real numbers $\epsilon = \{\epsilon_i\}_{i \in I}$, where I is an arbitrary collection of indexes, denumerable or not. Since it is a collection of real numbers, which may be constructed in the classical part of \mathfrak{Q} , we have that $Z(\epsilon)$. Intuitively, the elements ϵ_i represent the eigenvalues of a physical observable \hat{O} , that is, they are the values such that $\hat{O}|\varphi_i\rangle = \epsilon_i|\varphi_i\rangle$, with $|\varphi_i\rangle$ the corresponding eigenstates. Since observables are Hermitian operators, the eigenvalues are real numbers. Thus, we are justified in assuming that elements of ϵ are real numbers. Consider then the quasi-functions $f : \epsilon \rightarrow \mathcal{F}_p$, where \mathcal{F}_p is the quasi-set formed of all finite and pure quasi-sets (that is, finite quasi-sets whose only elements are indistinguishable m-atoms). Each of these f is a q-set of ordered pairs $\langle \epsilon_i, x \rangle$ with $\epsilon_i \in \epsilon$ and $x \in \mathcal{F}_p$. From \mathcal{F}_p we select those quasi-functions f which attribute a non-empty q-set only to a finite number of elements of ϵ , the image of f being \emptyset for the other cases. We call \mathcal{F} the quasi-set containing only these quasi-functions. Then, the quasi-cardinal of most of the q-sets attributed to elements of ϵ according to these quasi-functions is 0. Now, elements of \mathcal{F} are quasi-functions which we read as attributing to each ϵ_i a q-set whose quasi-cardinal we take to be the occupation number of this eigenvalue. We write these quasi-functions as $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}}$. According to the given intuitive interpretation, the levels $\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}$ are occupied. We say that if the symbol ϵ_{i_k} appears j -times, then the level ϵ_{i_k} has occupation number j . For example, the notation $f_{\epsilon_1 \epsilon_1 \epsilon_1 \epsilon_2 \epsilon_3}$ means that the level ϵ_1 has occupation number 3 while the levels ϵ_2 and ϵ_3 have occupation numbers 1. The levels that do not appear have occupation number zero. Another point to be remarked is that since the elements of ϵ are real numbers, we can take the standard ordering relation over the reals and order the indexes according to this ordering in the representation $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_m}}$. This will be important when we consider the cases for bosons and fermions.

The quasi-functions of \mathcal{F} provide the key to the solution to the problem of labeling states. Since we use pure quasi-sets as the images of the quasi-functions, there is simply no question of indexes for particles, for all that matters are the quasi-cardinals representing the occupation numbers. To make it clear that permutations change nothing, one needs only to notice that a quasi-function is a q-set of weakly ordered pairs.²⁶ Taking two of the pairs belonging to some quasi-function, let us say $\langle \epsilon_i, x \rangle, \langle \epsilon_j, y \rangle$, with both x and y non-empty, a permutation of particles would consist in changing elements from x with elements from y . However, by the unobservability of permutations theorem,²⁷ what we obtain after the permutation is a q-set indistinguishable from the one we began with. Remember also that a quasi-function attributes indistinguishable images to indistinguishable items; thus, the indistinguishable q-set resulting from the permutations

²⁶A weak ordered pair is a qset having just one element (that is, its cardinal is one). We cannot name such an element, for we need an identity to do that. SO, it can be taken as *one* element of a kind.

²⁷This theorem says that if we exchange an element of a qset by an indistinguishable one, the resulting qset turns to be indistinguishable from the original one.

will also be in the image of the same eigenvalue. To show this point precisely, we recall that by definition $\langle \epsilon_i, x \rangle$ abbreviates $[[\epsilon_i], [\epsilon_i, x]]$,²⁸ and an analogous expression holds for $\langle \epsilon_j, y \rangle$. Also, by definition, $[\epsilon_i, x]$ is the collection of all the items indistinguishable from ϵ_i or from x (taken from a previously given q-set). For this reason, if we permute x with x' , with $x \equiv x'$ we change nothing for $[\epsilon_i, x] \equiv [\epsilon_i, x']$. Thus, we obtain $\langle \epsilon_i, x \rangle \equiv \langle \epsilon_i, x' \rangle$ and the ordered pairs of the permuted quasi-function will be indiscernible (the same if there are no m-atoms involved). Thus, the permutation of indistinguishable elements does not produce changes in the quasi-functions.

7.2 A Vector Space Structure

Now, we wish to have a vector space structure to represent quantum states. To do that, we need to define addition and multiplication by scalars. Before we go on, we must notice that we cannot define these operations directly on the q-set \mathcal{F} , for there is no simple way to endow it with the required structure; our strategy here is to define \star (multiplication by scalars) and $+$ (addition of vectors) in a q-set whose vectors will be quasi-functions from \mathcal{F} to the set of complex numbers \mathbb{C} . Let us call C the collection of quasi-functions that assign to every $f \in \mathcal{F}$ a complex number. Once again, we select from C the sub-collection C_F of quasi-functions c such that every $c \in C_F$ attributes complex numbers $\lambda \neq 0$ for only a finite number of $f \in \mathcal{F}$. Over C_F , we can define a sum and a product by scalars in the same way as it is usually done with functions as follows.

Definition 7.1 *Let $\gamma \in \mathbb{C}$, and c, c_1 and c_2 be quasi-functions of C_F , then*

$$(\gamma \star c)(f) := \gamma(c(f))$$

$$(c_1 + c_2)(f) := c_1(f) + c_2(f)$$

The quasi-function $c_0 \in C_F$ such that $c_0(f) = 0$ for every $f \in \mathcal{F}$ acts as the null element for the sum operation. This can be shown as follows:

$$(c_0 + c)(f) = c_0(f) + c(f) = 0 + c(f) = c(f), \forall f. \quad (9)$$

With both the operations of sum and multiplication by scalars defined as above we have that $\langle C_F, \mathbb{C}, +, \star \rangle$ has the structure of a complex vector space, as one can easily check. Some of the elements of C_F have a special status though; if $c_j \in C_F$ are the quasi-functions such that $c_j(f_i) = \delta_{ij}$ (where δ_{ij} is the Kronecker symbol), then the vectors c_j are called the basis vectors, while the others are linear combinations of them. For notational convenience, we can introduce a new notation for the q-functions in C_F ; suppose c attributes a $\lambda \neq 0$ to some f , and 0 to every other quasi-function in \mathcal{F} . Then, we propose to denote c by λf . The basis quasi-functions will be denoted simply f_i , as one can check. Now, multiplication by scalar α of one of these quasi-functions, say λf_i can be read simply as $(\alpha \cdot \lambda) f_i$, and sum of quasi-functions λf_i and αf_i can be read as $(\alpha + \lambda) f_i$. What about the other quasi-functions in C_F ? We can extend this idea to them too, but with some care: if, for example c_0 is a quasi-function such that

²⁸We are leaving aside the subindices in this notation.

$c_0(f_i) = \alpha$ and $c_0(f_j) = \lambda$, attributing 0 to every other quasi-function in \mathcal{F} , then c_0 can be seen as a linear combination of quasi-functions of a basis; in fact, consider the basis quasi-functions f_i and f_j , (this is an abuse of notation, for they are representing quasi-functions in C_F that attribute 1 to each of these quasi-functions). The first step consists in multiplying them by α and λ , respectively, obtaining αf_i and λf_j (once again, this is an abuse, for these are quasi-functions in C_F that attribute the mentioned complex numbers to f_i and to f_j). Now, c_0 is in fact the sum of these quasi-functions, that is, $c_0 = \alpha f_i + \lambda f_j$, for this is the function which does exactly what c_0 does. One can then extend this to all the other quasi-functions in C_F as well.

7.3 Inner Products

The next step in our construction is to endow our vector space with an inner product. This is a necessary step for we wish to calculate probabilities and mean values. Following the idea proposed in [11], we introduce two kinds of inner products, which lead us to two Hilbert spaces, one for bosons and another for fermions. We begin with the case for bosons.

Definition 7.2 Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_1 \epsilon_2 \dots \epsilon_n}$ and $f_{\epsilon'_1 \epsilon'_2 \dots \epsilon'_m}$ two basis vectors (as discussed above), then

$$f_{\epsilon_1 \epsilon_2 \dots \epsilon_n} \circ f_{\epsilon'_1 \epsilon'_2 \dots \epsilon'_m} := \delta_{nm} \sum_p \delta_{i_1 p'_1} \delta_{i_2 p'_2} \dots \delta_{i_n p'_n}. \quad (10)$$

Notice that this sum is extended over all the permutations of the index set $i' = (i'_1, i'_2, \dots, i'_n)$; for each permutation p , $p'_i = (p'_1, p'_2, \dots, p'_n)$.

For the other vectors, the ones that can be seen as linear combinations in the sense discussed above, we have

$$\left(\sum_k \alpha_k f_k \right) \circ \left(\sum_k \alpha'_k f'_k \right) := \sum_{kj} \alpha_k^* \alpha'_j (f_k \circ f'_j), \quad (11)$$

where α^* is the complex conjugate of α . Now, let us consider fermions. As remarked above in page 23, the order of the indexes in each $f_{\epsilon_1 \epsilon_2 \dots \epsilon_n}$ is determined by the canonical ordering in the real numbers. Thus, we define another \bullet inner product as follows, which will do the job for fermions.

Definition 7.3 Let δ_{ij} be the Kronecker symbol and $f_{\epsilon_1 \epsilon_2 \dots \epsilon_n}$ and $f_{\epsilon'_1 \epsilon'_2 \dots \epsilon'_m}$ two basis vectors, then

$$f_{\epsilon_1 \epsilon_2 \dots \epsilon_n} \bullet f_{\epsilon'_1 \epsilon'_2 \dots \epsilon'_m} := \delta_{nm} \sum_p \sigma_p \delta_{i_1 p'_1} \delta_{i_2 p'_2} \dots \delta_{i_n p'_n} \quad (12)$$

where: $\sigma_p = 1$ if p is even and $\sigma_p = -1$ if p is odd.

This definition can be extended to linear combinations as in the previous case.

7.4 Fock spaces using \mathcal{Q} -spaces

We begin with a definition to simplify the notation. For every function $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$ in \mathcal{F} , we put

$$\alpha|\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}\rangle := \alpha f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$$

Note that this is a slightly modified version of the standard notation. We begin with the case of bosons.

Suppose a normalized vector $|\alpha\beta\gamma\dots\rangle$, where the norm is taken from the corresponding inner product. Let ζ stand for an arbitrary collection of indexes. We define $a_\alpha^\dagger|\zeta\rangle \propto |\alpha\zeta\rangle$ in such a way that the proportionality constant satisfies $a_\alpha^\dagger a_\alpha|\zeta\rangle = n_\alpha|\zeta\rangle$. From this it will follow, as usual, that:

$$((\zeta|a_\alpha^\dagger)(a_\alpha|\zeta)) = n_\alpha.$$

Definition 7.4 $a_\alpha|\dots n_\alpha \dots\rangle := \sqrt{n_\alpha}|\dots n_\alpha - 1 \dots\rangle$

On the other hand,

$$a_\alpha a_\alpha^\dagger|\dots n_\alpha \dots\rangle = K \sqrt{n_\alpha + 1}|\dots n_\alpha \dots\rangle,$$

where K is a proportionality constant. Applying a_α^\dagger again, we have

$$a_\alpha^\dagger a_\alpha a_\alpha^\dagger|\dots n_\alpha \dots\rangle = K^2 \sqrt{n_\alpha + 1}|\dots n_\alpha + 1 \dots\rangle.$$

Using the fact that $a_\alpha^\dagger a_\alpha|\zeta\rangle = n_\alpha|\zeta\rangle$, we have that

$$(a_\alpha^\dagger a_\alpha) a_\alpha^\dagger|\dots n_\alpha \dots\rangle = \sqrt{n_\alpha + 1} K |\dots n_\alpha + 1 \dots\rangle.$$

So, $K = \sqrt{n_\alpha + 1}$. Then, we have

Definition 7.5 $a_\alpha^\dagger|\dots n_\alpha \dots\rangle := \sqrt{n_\alpha + 1}|\dots n_\alpha + 1 \dots\rangle$.

From this definition, with additional computations, we obtain $(a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha)|\psi\rangle = \delta_{\alpha\beta}|\psi\rangle$. In our language, this means the same as

$$[a_\alpha; a_\beta^\dagger] = \delta_{\alpha\beta} I.$$

In an analogous way, it can be shown that

$$[a_\alpha; a_\beta] = [a_\alpha^\dagger; a_\beta^\dagger] = 0.$$

So, the bosonic commutation relation is the same as in standard Fock space formalism.

For fermionic states, we use the antisymmetric product “ \bullet .” We begin by defining the creation operator C_α^\dagger .

Definition 7.6 *If ζ is a collection of indexes of non-null occupation numbers, then $C_\alpha^\dagger := \alpha|\zeta\rangle$*

If α is in ζ , then $|\alpha\zeta\rangle$ is a vector of null norm. This implies that $(\psi|\alpha\zeta) = 0$, for every ψ . It follows that systems in states of null norm have no probability of being observed. Furthermore, their addition to another vector does not contribute to any observable difference. To take the situation into account, we have the following definition.

Definition 7.7 *Two vectors $|\phi\rangle$ and $|\psi\rangle$ are similar if the difference between them is a linear combination of null norm vectors. We denote similarity of $|\phi\rangle$ and $|\psi\rangle$ by $|\phi\rangle \cong |\psi\rangle$.*

Using the definition of C_α^\dagger we can describe what is the effect of C_α over vectors, namely

$$(\zeta|C_\alpha := (\alpha\zeta|.$$

Then, for any vector $|\psi\rangle$,

$$(\zeta|C_\alpha|\psi) = (\alpha\zeta|\psi) = 0$$

for $\alpha \in \zeta$ or $(\psi|\alpha\zeta) = 0$. Then, if $|\psi\rangle = |0\rangle$, then $(\zeta|C_\alpha|0) = (\alpha\zeta|0) = 0$. So, $C_\alpha|0\rangle$ is orthogonal to any vector that contains α , and also to any vector that does not contain α , so that it is a linear combination of null norm vectors. So, we can put by definition that $\vec{0} := C_\alpha|0\rangle$. In an analogous way, if $\sim \alpha$ denotes that α has occupation number zero, then we can also write $C_\alpha|(\sim \alpha) \dots\rangle = \vec{0}$, where the dots mean that other levels have arbitrary occupation numbers.

Now, using our notion of similar vectors, we can write $C_\alpha|0\rangle \cong \vec{0}$ and $C_\alpha|(\sim \alpha) \dots\rangle \cong \vec{0}$. The same results are obtained when we use \cong and the sign of identity. By making $|\psi\rangle = |\alpha\rangle$, we have $(\zeta|C_\alpha|\alpha) = (\alpha\zeta|\alpha) = 0$ in every case, except when $|\zeta\rangle = |0\rangle$. In that case, $(0|C_\alpha|\alpha) = 1$. Then, it follows that $C_\alpha|\alpha\rangle \cong 0$. In an analogous way, we obtain $C_\alpha|\alpha\zeta\rangle \cong |(\sim \alpha)\zeta\rangle$ when $\alpha \notin \zeta$. In the case $\alpha \in \zeta$, $|\alpha\zeta\rangle$ has null norm, and so, for every $|\psi\rangle$:

$$(\alpha\zeta|C_\alpha^\dagger|\psi) = (\alpha\zeta|\alpha\psi) = 0.$$

It then follows that

$$(\psi|C_\alpha|\alpha\zeta) = 0,$$

so that $C_\alpha|\alpha\zeta\rangle$ has null norm too.

Now we calculate the anti-commutation relation obeyed by the fermionic creation and annihilation operators. We begin calculating the commutation relation between C_α and C_β^\dagger . We do that by studying the relationship between $|\alpha\beta\rangle$ and $|\beta\alpha\rangle$. Let us consider the sum $|\alpha\beta\rangle + |\beta\alpha\rangle$. The product of this sum with any vector distinct from $|\alpha\beta\rangle$ is null. For the product with $|\alpha\beta\rangle$ we obtain $(\alpha\beta|[|\alpha\beta\rangle + |\beta\alpha\rangle]) = (\alpha\beta|\alpha\beta) + (\alpha\beta|\beta\alpha)$. By definition, this is equal to $\delta_{\alpha\alpha}\delta_{\beta\beta} - \delta_{\alpha\beta}\delta_{\beta\alpha} + \delta_{\alpha\beta}\delta_{\alpha\alpha} - \delta_{\alpha\alpha}\delta_{\beta\beta}$. This is equal to $1 - 0 + 0 - 1 = 0$.

The same conclusion holds if we multiply the sum $|\alpha\beta\rangle + |\beta\alpha\rangle$ by $|\beta\alpha\rangle$. It then follows that $|\alpha\beta\rangle + |\beta\alpha\rangle$ is a linear combination of null norm vectors, which we denote by $|nn\rangle$, so that

$$|\alpha\beta\rangle = -|\beta\alpha\rangle + |nn\rangle.$$

Given that, we can calculate

$$C_\alpha^\dagger C_\beta^\dagger|\psi) = |\alpha\beta\psi) = -|\beta\alpha\psi) + |nn) = -C_\beta^\dagger C_\alpha^\dagger|\psi) + |nn).$$

From this it follows that $\{C_\alpha^\dagger; C_\beta^\dagger\}|\psi\rangle = |nn\rangle$. We do not lose generality by setting $\{C_\alpha^\dagger; C_\beta^\dagger\}|\psi\rangle = 0$. In an analogous way we conclude that

$$\{C_\alpha; C_\beta\}|\psi\rangle = 0.$$

Now we calculate the commutation relation between C_α and C_β^\dagger . There are some cases to be considered. We first assume that $\alpha \neq \beta$. If $\alpha \notin \psi$ or $\beta \in \psi$ then

$$\{C_\alpha; C_\beta^\dagger\}|\psi\rangle \approx \vec{0}.$$

If $\alpha \in \psi$ and $\beta \notin \psi$, assuming that α is the first symbol in the list of ψ , then $\{C_\alpha; C_\beta^\dagger\}|\psi\rangle = C_\alpha|\beta\psi\rangle + C_\beta^\dagger|\psi(\sim \alpha)\rangle \cong -|\beta\psi(\sim \alpha)\rangle + |\beta\psi(\sim \alpha)\rangle = \vec{0}$. If $\alpha = \beta$ and $\alpha \in \psi$, then $\{C_\alpha; C_\alpha^\dagger\}|\psi\rangle = C_\alpha|\alpha\psi\rangle + C_\alpha^\dagger|\psi(\sim \alpha)\rangle \cong \vec{0} + |\psi\rangle = |\psi\rangle$. If $\alpha = \beta$ and $\alpha \notin \psi$, then $\{C_\alpha; C_\alpha^\dagger\}|\psi\rangle = C_\alpha|\alpha\psi\rangle + C_\alpha^\dagger|\psi(\sim \alpha)\rangle \cong |\psi\rangle + \vec{0} = |\psi\rangle$. In any case, we recover $\{C_\alpha; C_\alpha^\dagger\}|\psi\rangle \cong \delta_{\alpha\beta}|\psi\rangle$. So, we can put

$$\{C_\alpha; C_\alpha^\dagger\} = \delta_{\alpha\beta}.$$

It then follows that the commutation properties in \mathfrak{Q} -spaces are the same as in traditional Fock spaces.

Using this formalism, we can adapt all the developments done in [26, Chap.7] and [27, Chap.20] for the number occupation formalism. However, contrary to what happens in these books, no previous (even unconscious) assumptions about quantum objects' individuality is taken into account.

8 Conclusions

It is an exciting question to ask if we need to change logic every time we find difficulties with the classical one. Are there other ways to circumvent the problems, such as in the quantum case, using the tricks mentioned above, or choosing an alternative interpretation? This question makes sense. However, we think that every theory, even a mathematical one, starts from metaphysical hypotheses, even if not made explicit. We have stated above that classical logic, standard mathematics, and classical physics were developed with the classical enclosing world in our minds. This world is one of individuals that have an identity. So, two of those individuals cannot possibly be different.

Nevertheless, quantum mechanics brought us a different world, a world with no proper names. In the quantum world, objects are (in most cases) precisely alike, and permutations between objects of the same kind do not lead to any physical differences. Here we emphasize that it is not that these are not *measurable* differences; *they are no differences at all*. So, we arrive at the following conclusions.

1. Indistinguishability is essential in quantum mechanics, regardless of interpretation. In our opinion, it should be placed at an equal level of importance in quantum foundations to concepts such as entanglement, contextuality, and nonlocality.

2. Ontological and epistemic aspects matter. Any physical theory is grounded in interpretations due to the possibility of associating different world views (or metaphysics) to a theory. Parodying Poincaré, we can say that physics is (also) a domain where we give the same name to distinct things.²⁹
3. Since mathematics and logic need to reflect the assumed metaphysical aspects (we could speak in terms of ontology), quantum mechanics' formalism and physical theories should do the same.

Let us expand on this last point with an example involving logic. It is common to say that in order to obtain intuitionistic logic, it is enough to drop the excluded middle law from the axioms of classical logic. From a purely formal point of view, this is correct. However, logic is not only syntax. It also involves semantic aspects and even pragmatic ones (making references to who uses the logic and why). Let us consider semantics. Although classical and intuitionistic logic differs syntactically just by one axiom, semantically, they are much different. Classical propositional logic can be described through truth-tables; intuitionistic logic cannot. In classical logic, any proposition is either true or false, yet we may not know what the case is; in intuitionistic logic, the notions of true and false are different. In this logic, a proposition p is true if there is a “process” to get it, and false if a process for obtaining p leads to a contradiction. Other differences can be pointed out. For instance, in classical logic, something exists if its nonexistence creates a contradiction. In intuitionistic logic, something exists if it can be created by our imagination.

This example shows that in order to consider a logic, semantical aspects must at least be considered. Of course, this is true also with physical theories. Otherwise, we risk having a purely mathematical theory. However, what corresponds to semantics in the quantum case? We chose interpretations because quantum mechanics, as Yuri Manin wrote, “does not really have its own language” [25, p. 84]. At least not yet. Indeed, the standard formalism grounded on Hilbert spaces makes use of the language of standard functional analysis, which presupposes classical mathematics and logic, with all the problems seen before (in regarding quantum phenomena). A proper language should reflect the indiscernibility of quanta from the start, without tricks!

As we showed in this paper, such a correct language can be constructed. In this paper, we examined content and context in quantum physics. We provided examples of context for the classical and quantum realms and argued that the quantum situation is fundamentally different. Furthermore, we reasoned that context-dependency in the quantum world is intrinsically connected to the lack of identity. Thus, the non-identity of individuals is an essential feature of the quantum world. Since the standard mathematics used in physics does not exactly allow for objects who lack identity, i.e., indistinguishable objects, we advocated for using a different mathematical structure in physics: quasi-set theory. Quasi-set theory includes standard mathematics in it but also contains indistinguishable objects. We believe that recreating quantum physics in terms of quasi-set theory and its underlying logic would result in thinking closer to a more

²⁹Poincaré was referring to mathematics: “mathematics is the art of giving the same name to distinct things” — look at [38]. Of course, he spoke within the framework of axiomatized mathematical theories, able to have different models.

reasonable ontology for the quantum world than currently available ontologies. This way of thinking may lead to exciting insights into quantum ontologies and fundamental physical principles that define quantum mechanics.

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