Classical Fields, Bell's inequalities, and the quantum limit

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Abstract

In this paper, we use a homodyne detection of a classical field to violate Bell's inequalities. This violation is achieved with local random variables that are continuous, which does not preclude the existence of a joint probability distribution. This result and meanings are discussed.

We dedicate this article to Professor Francisco Doria (FAD) on the occasion of his 70th birthday. We believe it is apt to honor him with this article for several reasons. First, this paper is an update of a previous article co-authored with Pat Suppes (available on arXiv:quant-ph/9606019). Pat and Doria were good friends, and Pat would have been delighted to contribute to this volume. Second, Doria spent a one-year sabbatical leave in the '90s as a Fulbright scholar visiting Pat at Stanford University's Institute for Mathematical Studies in the Social Sciences (IMSSS), which Pat founded and directed. This contact between Pat and Doria led to ASS and JAB spending some time at the IMSSS, where this paper's ideas germinated. Third, this paper exemplifies the type of research that Doria does: foundation issues in physics and mathematics that lead to different insights. Finally, both ASS and JAB owe their passion for science and mathematics to Doria's guidance and tutoring. So, this work would not have been written without his early support and involvement.

Most of the outcomes of this paper are not new. As mentioned above, the main result of a violation of Bell's inequalities with classical fields was present in our '96 manuscript with Pat Suppes. However, that article was never published. Furthermore, in recent years much interest appeared in connection with violations of quantum inequalities with classical systems (see [26] and references therein). In a certain sense, our paper came to this area too early when few worked on it. Thus, we believe it is as relevant today as it was more than 20 years ago.

1 Introduction

Quantum theory is bizarre. This strangeness comes mostly from assigning properties to a quantum system, famously exemplified by the Einstein-Bohr dialogues. Bohr believed that a property of physical systems only existed if an observer made an actual measurement of this property. To Bohr, talking about properties that were not measured was nonsensical. Einstein, on the other hand, believed that properties had a reality independent of the observer. To him, the reality of a system's property did not require performing an actual measurement. Since quantum theory did not provide a way to talk about unobserved quantities, the observer-independent reality, Einstein concluded that quantum theory was incomplete and that a complete theory should be developed. Such theories that extended quantum mechanics are known as *hidden-variable theories*.

Perhaps the most persuasive argument in favor of a hidden variable theory was that of Einstein, Podolsky, and Rosen (EPR) [19]. Consider two spin-1/2 particles are produced in the singlet state[1]

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(| + -\rangle - | - +\rangle \right),$$

where $|+\rangle$ and $|-\rangle$ are the eigenvectors of the spin operator (say, in direction $\mathbf{z}$). One of the particles is sent to Alice's laboratory (A) and the other to Bob's lab (B). EPR noticed that if Alice measured the spin to be "+," she could immediately infer that Bob's measurement of the spin would be "−." To EPR, this meant that we could infer the value of the particle's spin at Bob's without interacting with it, i.e., without disturbing it, since we should rule out non-local interactions between particles in Alice and Bob's labs, as those labs can be placed as far as we want. Thus, EPR concluded that quantum theory was incomplete, as there were elements of reality (the value of spin) that could be inferred without any measurement. To complete quantum mechanics, one would need a hidden variable theory.

There are several obstacles to developing hidden-variable theories. For example, without resorting to EPR's argument, a question remains as to whether we can simultaneously assign values to complementary properties, such as momentum and position. In a famous paper, Kochen and Specker [23] showed that if we try to do so, we reach a contradiction, unless we assume that a property changes with the experimental context. For example, imagine a quantum system with four properties, A, B, C, and D. Further assume that

[1] The version of the argument presented here is Bohm's, and not EPR's [9]. EPR used momentum and position as correlated variables instead of spin.

we can create experiments where we can only measure the following variables simultaneously: A and C; A and D; B and C; and B and D. In this example, A shows up in two measuring contexts: A in the context of C and A in the context of D. Given that an experimenter can choose at will which context to measure, for the property A to be independent of the experimenter's capricious choices, it needs to be independent of the experimental context. However, Kochen and Specker proved that this idea that a property, in this case, A, is independent of those two contexts, leads to contradictions. In other words, Kochen and Specker showed that any hidden-variable theory has to be contextual.

Another important obstacle to hidden-variable theories was shown in 1963 by John Bell [6, 7]. In his paper, Bell discussed Bohm's setup for the Einstein-Podolsky-Rosen's (EPR) Gedankenexperiment [9, 19]. By assuming EPR's criteria of realism and locality, Bell derived a set of inequalities that any local hidden-variable theory had to satisfy, known as Bell's inequalities. Bell then proceeded to show that, for certain situations, quantum mechanics violated his inequalities. In other words, if EPR was right that quantum theory was incomplete and should be substituted by a local hidden-variable theory, then Bell determined that certain predictions of quantum mechanics had to be wrong. In 1982 Alain Aspect and collaborators confirmed that correlated quantum systems indeed violate Bell's inequalities. Thus, local hidden-variable theory are wrong [4, 3, 2].

Bell's assumptions are considered equivalent to an underlying physical reality, with added locality conditions. As such, they are equivalent to the existence of a joint probability distribution for all possible outcomes of an experiment in all possible contexts [32, 20]. What this means is that, for particular conjunction of properties, one cannot assign a probability value consistent with the observed marginals[2]. This is equivalent to saying that properties are contextual (see [15] and references therein).

Because quantum non-locality and contextuality are among the most disturbing aspects of the theory, it was commonly believed that any classical system satisfies Bell's inequalities. In this paper, we show that classical fields *do not* satisfy Bell's inequalities. Hence classical fields, e.g., electromagnetic fields, are not Bell-type hidden variables. We do this by using a simple experimental setup proposed by Tan *et al.* [33, 34, 35] for single photons. We then reinterpret this setup for classical electromagnetic fields with randomized phase. For this proposed experiment, we derive from the classical field properties a violation of Bell's inequalities [6, 7, 8, 32], with, at the same time, locality being preserved in a sense to be made precise.

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[2] An alternative is to use extended probabilities. See [30, 31, 17, 18, 16, 15] for examples using non-monotonic upper probabilities as well as (signed) negative probabilities.

published. Furthermore, in recent years much interest appeared in connection with violations of quantum inequalities with classical systems [24, 1, 11, 22, 28, 5, 27, 26, 25]. In a certain sense, our paper came to this area too early when nobody was working on it. Thus, we believe it is as relevant today as it was more than 20 years ago.

2 Experimental Setup

As mentioned above, we use a similar experimental scheme to that of Tan, Walls, and Collett [34]. In their experiment, Tan et al. used a single-photon source and a beam splitter to create an entanglement between a single photon and the vacuum in two separate beams. Each beam's phase was measured via two homodyne detections [14] with same frequency but fixed phases θ_1 and θ_2. They showed that for two different homodyne phases θ_1 and θ_2, the detection statistics showed correlations between each detector that violated Bell's inequalities. Equally important, according to Tan et al., a classical coherent field would not violate Bell's inequalities. Here we argue that their conclusion about classical fields is not correct, as they assumed a fixed phase for the coherent state. In what follows, we show that a uniformly distributed phase in the interval $[0, 2\pi]$ yields a correlation between detectors that violates Bell's inequalities.

Our experimental scheme uses three *classical* coherent sources: $\alpha_1(\theta_1)$, with fixed phase θ_1; $\alpha_2(\theta_2)$, with fixed phase θ_2; and $u(\theta)$, with a unknown stochastic phase θ. The geometry of the setup is shown in Figure 1. The experimental configuration has two homodyne detections, (D_1, D_2) being one and (D_3, D_4) the other, such that the measurements are sensitive to phase changes in $u(\theta)$. Similarly to Tan et al.'s experiment, in Figure 1, BS, BS_1, and BS_2 are beam splitters that reflect 50% of the incident electromagnetic field and let 50% of it pass. When the electromagnetic field is reflected, the mirrors add a phase of $\pi/2$ to the field, while no phase is added when the field passes through BS, BS_1, or BS_2. We will look for correlations between the pairs of detectors (D_1, D_2) and (D_3, D_4).

To compute the correlation functions, we first define the *continuous* random variables in terms of which we derive the Bell-type correlations. On this matter we shall be as explicit as possible. Associated to the source $u(\theta)$ at D_1 is the random variable $U_1(t)$, whose value at t is just the value of the classical field at D_1, namely,

$$U_1(t) = \frac{1}{4}\beta \cos(\omega t + \theta + \pi/2), \tag{1}$$

where β is the amplitude of the field at the source, θ is the unknown phase and $\pi/2$ is a phase gained when u is reflected at BS_2.

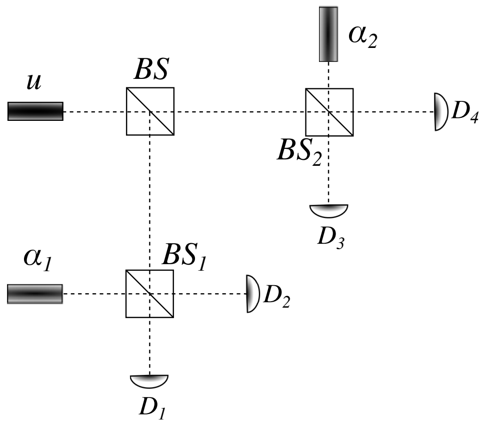

Figure 1: Experimental configuration. A laser u with stable phase θ impinges on a 50:50 beam splitter (BS). The two equal-intensity beams then impinge on two additional 50:50 beam splitters, BS_1 and BS_2. At BS_1 and BS_2, the beam from u is combined with lasers α_1 and α_2 (with half the intensity of u), respectively, with phases θ_1 and θ_2. Detectors $D_1, \ldots, D_4$ register the intensity of the fields at each arm of the beam splitters.

Probability enters initially by using the time average to compute the expectation of $U_1(t)^2$

$$U_1^2 = \langle U_1(t)^2 \rangle = \langle [\frac{1}{4}\beta \cos(\omega t + \theta + \pi/2)]^2 \rangle,$$

which is just the standard intensity, but here we treat it probabilistically. In the above expression, $\langle f(t) \rangle$ is defined as the temporal expectation given by

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt. \tag{2}$$

We emphasize that (2) is expressed in terms of a limit as T goes to infinity for mathematical simplicity, but in practice it suffices that T is large enough to stabilize the expectation values (e.g., if $T \gg 1/\omega$). In a similar fashion, associated to the source $\alpha_1(\theta_1)$ at D_1 is the random variable $A_1(t)$,

$$A_1(t) = \frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2)$$

and thus

$$A_1^2 = \langle A_1(t)^2 \rangle = \left\langle \left[\frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2) \right]^2 \right\rangle.$$

At D_1, the total field is the random variable $F_1(t) = U_1(t) + A_1(t)$. So, the intensity of the total field at D_1 is just the second moment of $F_1(t)$, i.e.,

$$\begin{aligned} I_1(\theta) &= F_1^2 = \langle F_1(t)^2 \rangle = \langle (U_1(t) + A_1(t))^2 \rangle \\ &= \langle U_1(t)^2 \rangle + 2\langle (U_1(t) A_1(t)) \rangle + \langle A_1(t)^2 \rangle, \end{aligned}$$

where we used θ as an argument for I_1 to make it explicit that it depends on θ. We can see that the cross moment in the expression above is the classical interference term.

We can compute I_1 directly from the expression for $U_1(t)$ and $A_1(t)$ in the following way

$$\begin{aligned} I_1(\theta) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\frac{1}{2}\alpha \cos(\omega t + \theta_1 + \pi/2) + \right. \\ & \left. \frac{1}{4}\beta \cos(\omega t + \theta + \pi/2) \right]^2 dt, \end{aligned}$$

which yields

$$I_1(\theta) = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha\beta\cos(\theta - \theta_1) + \frac{1}{8}\alpha^2. \tag{3}$$

In similar fashion, we can compute for the other three detectors,

$$I_2(\theta) = \frac{1}{32}\beta^2 - \frac{1}{8}\alpha\beta\cos(\theta - \theta_1) + \frac{1}{8}\alpha^2, \tag{4}$$

$$I_3(\theta) = \frac{1}{32}\beta^2 - \frac{1}{8}\alpha\beta\sin(\theta - \theta_2) + \frac{1}{8}\alpha^2, \tag{5}$$

and

$$I_4(\theta) = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha\beta\sin(\theta - \theta_2) + \frac{1}{8}\alpha^2. \tag{6}$$

The intensities obtained are conditional on θ. To obtain the unconditional intensities we assume a uniform distribution for θ and integrate the expressions for all possible values of θ. Not only is θ unknown, but the phase would vary randomly for repeated runs of the experiment. If θ were a coherent source with fixed θ, Bell's inequalities would not be violated [33].

The unconditional intensities I_1, I_2, I_3, and I_4 for the detectors D_1, D_2, D_3, and D_4 are

$$I_1 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{7}$$

$$I_2 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{8}$$

$$I_3 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2, \tag{9}$$

$$I_4 = \frac{1}{32}\beta^2 + \frac{1}{8}\alpha^2. \tag{10}$$

We can see from (7)–(10) that the intensities are the same for all detectors, and are similar to those given by Walls and Milburn [35] in the case of a classical source. This is expected, since (7)–(10) express the concept that the intensity at the detectors is the sum of the intensities from the two sources.

We now start computing the covariance between intensities in the homodyne detectors. The covariance we are interested in is between $(I_1 - I_2)$ and $(I_3 - I_4)$.

$$\begin{aligned}\mathrm{Cov}\,(I_1 - I_2, I_3 - I_4) &= \frac{1}{2\pi}\int_0^{2\pi} [(I_1(\theta) - I_2(\theta)) \times (I_3(\theta) - I_4(\theta))]d\theta \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} (I_1(\theta) - I_2(\theta))\,d\theta \\ &\quad \times \frac{1}{2\pi}\int_0^{2\pi} (I_3(\theta) - I_4(\theta))\,d\theta. \end{aligned} \tag{11}$$

It is straightforward to show from (3)–(6) and (11) that

$$\mathrm{Cov}\,(I_1 - I_2, I_3 - I_4) = -\frac{1}{32}\beta^2\alpha^2 \sin(\theta_1 - \theta_2)\,.$$

In order to compute the correlation we have to know the variance of the random variables $(I_1 - I_2)$ and $(I_3 - I_4)$, which are defined in a standard way as

$$\begin{aligned}\mathrm{Var}(I_1 - I_2) &= \frac{1}{2\pi}\int_0^{2\pi} (I_1(\theta) - I_2(\theta))^2 d\theta - \left[\frac{1}{2\pi}\int_0^{2\pi} (I_1(\theta) - I_2(\theta))d\theta\right]^2 \\ &= \frac{1}{32}\beta^2\alpha^2\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(I_3 - I_4) &= \frac{1}{2\pi}\int_0^{2\pi} (I_3(\theta) - I_4(\theta))^2 d\theta - \left[\frac{1}{2\pi}\int_0^{2\pi} (I_3(\theta) - I_4(\theta))d\theta\right]^2 \\ &= \frac{1}{32}\beta^2\alpha^2.\end{aligned}$$

Finally, we are in a position to compute the correlation between the two random variables $A \equiv (I_1 - I_2)$ and $B \equiv (I_3 - I_4)$. This is done in a standard way by just dividing the covariance by the square-root of the variances [29]:

$$\rho(A, B) = \frac{\mathrm{Cov}(A, B)}{\sqrt{\mathrm{Var}(A)\ \mathrm{Var}(B)}},$$

and we have the following expression for the correlation

$$\rho(A, B) = -\sin(\theta_1 - \theta_2),$$

which we may rewrite as

$$\rho(A(\theta_1), B(\theta_2)) = -\sin(\theta_1 - \theta_2), \tag{12}$$

to make explicit the dependency of A and B on homodyning phase angles θ_1 and θ_2. This correlation is the same as the well-known correlations for Bell's setup [6].

So, we are now in a position to show that we can violate Bell-type inequalities. We may now choose angles θ_1, θ_2, θ_1', and θ_2' such that we obtain at once, for the four correlations $\rho(\theta_1, \theta_2)$, $\rho(\theta_1, \theta_2')$, $\rho(\theta_1', \theta_2)$ and $\rho(\theta_1', \theta_2')$ a violation of Bell's inequalities in the form due to Clauser, Horne, Shimony, and Holt (CHSH) [12, 13], by choosing the four angles such that

$$\theta_1 - \theta_2 = \theta_1' - \theta_2' = 60^o,$$

$$\theta_1 - \theta_2' = 30^o,$$

$$\theta_1' - \theta_2 = 90^o.$$

In particular,

$$\rho(\theta_1, \theta_2) - \rho(\theta_1, \theta_2') + \rho(\theta_1', \theta_2) + \rho(\theta_1', \theta_2') = -\frac{\sqrt{3}}{2} + \frac{1}{2} - 1 - \frac{\sqrt{3}}{2} < -2.$$

However, in the case of continuous random variables, which is what we have in the present context for intensity, or differences of intensity, failure to satisfy Bell's inequalities in the CHSH form does not imply that there can be no joint distribution of the four random variables compatible with the four given correlations. It is easy to show that for selected values of the two missing correlations, there does, for this example, exist a joint probability of the four random variables compatible with the four given correlations.

3 Measurement and Photon Counts

Because classical field theory is a deterministic theory, our introduction of expectations and probabilities might be questioned. Our response is that the strength of a classical field at a space-time point cannot be measured, as was emphasized long ago by Bohr and Rosenfeld [10]. As they pointed out, classical field strength cannot be represented by true point functions, but by average values over space-time regions. This is exactly what we have done in introducing random variables and their expectations. The casual reader might claim that we should do an analysis of coincidence counts with photocounters. This makes no sense in the case of classical fields, where the number of photons arriving at the same time at each detector is incredibly large. What makes sense is not discrete but continuous measurement of intensity.

Despite that, we are going to use the previous result to model discrete photon counts, and show that this time it does not in such a way that they violate Bell's inequalities. For this, we define two new discrete random variables $X = \pm 1$ and $Y = \pm 1$. These random variables correspond to nearly simultaneous correlated counts at the detectors, and are defined in the following way.

$$X = \begin{cases} +1 & \text{if detector } D_1 \text{ triggers a count} \\ -1 & \text{if detector } D_2 \text{ triggers a count} \end{cases}$$

$$Y = \begin{cases} +1 & \text{if detector } D_3 \text{ triggers a count} \\ -1 & \text{if detector } D_4 \text{ triggers a count.} \end{cases}$$

To compute the expectation of X and Y we use the stationarity of the process and do the following. First, let us note that

$$I_1 - I_2 = N_X \cdot P(X = 1) - N_X \cdot P(X = -1),$$

where N_X is the expected total number of photon counts at D_1 and D_2 and $P(X = \pm 1)$ is the probability that the random variable X has values ± 1. The same relation holds for

$$I_3 - I_4 = N_Y \cdot P(Y = 1) - N_Y \cdot P(Y = -1).$$

To simplify we put as a symmetry condition that $N_X = N_Y = N$, i.e., the expected number of photon counts at each homodyne detector is the same. But we know that

$$I_1 + I_2 = N \cdot P(X = 1) + N \cdot P(X = -1) = N,$$

and

$$I_3 + I_4 = N \cdot P(X = 1) + N \cdot P(X = -1) = N.$$

Then we can conclude from equations (3)–(6), assuming maximum visibility, that

$$E_d(X|\theta) = \frac{I_1 - I_2}{I_1 + I_2} = \cos(\theta - \theta_i),$$

$$E_d(Y|\theta) = \frac{I_3 - I_4}{I_3 + I_4} = \sin(\theta - \theta_j),$$

where E_d represents the expected value of the counting random variable. It is clear that if θ is uniformly distributed we have at once:

$$E(X) = E_\theta(E_d(X|\theta)) = 0, \tag{13}$$

$$E(Y) = E_\theta(E_d(X|\theta)) = 0. \tag{14}$$

We can now compute $\mathrm{Cov}(X,Y)$. Note that

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E(XY) - E(X)E(Y) \\ &= E_\theta(E_d(XY|\theta)) - E_\theta(E_d(X|\theta))E_\theta(E_d(Y|\theta))\end{aligned}$$

and so

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \frac{1}{2\pi}\int_0^{2\pi} E_d(XY|\theta)d\theta \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} E_d(X|\theta)d\theta \times \frac{1}{2\pi}\int_0^{2\pi} E_d(Y|\theta)d\theta.\end{aligned}$$

In order to compute the covariance, we also use the conditional independence of X and Y given θ, which is our locality condition:

$$E_d(XY|\theta) = E_d(X|\theta)E_d(Y|\theta),$$

because given θ, the expectation of X depends only on θ_i, and of Y only on θ_j. Then, it is easy to see that

$$\rho(X,Y) = \mathrm{Cov}(X,Y) = -\sin(\theta_i - \theta_j). \tag{15}$$

The correlation equals the covariance, since X and Y are discrete ± 1-valued random variables with zero mean, as shown in (13) and (14), and so $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$. It follows at once from (15) that for a given set of θ_i's and θ_j's Bell's inequalities are violated.

However, there is an important detail underlying our computations above. When we are looking for correlations, we are only considering the cases where an "observation" is made on both detectors, i.e., we have coincidence counts for X and Y. However, to have a coincidence count, there are two assumptions that were tacitly introduced. First, for the given observation time window, the assumption is that we have a detection in one of the photodetectors. For the hidden-variable model that associates with field intensity a probability of photo detection, there should be a non-zero probability that no "detection" happens during a time window. This was not included in our model. Second, even if we accounted the no-detection event, and conditioned on coincidence detections, we would have to use a non-enhancement hypothesis [21] to compute the correlations in the way we are doing.

4 Final Remarks

The experiment proposed in [34] supposes a single photon source that is split into the two homodyne detectors. Tan *et al.* also analyze the classical case and get no violation of Bell's inequalities. However, they assume a weak coherent source with randomized phase as the classical analogue of their single photon source. This would be equivalent to having a classical thermal source, where coherence would not be a strong feature. In our experiment we suppose that this source is not only classical, i.e., with high intensity, but also that it is coherent with the phase unobservable and varying randomly on repeated runs. The different source used here, as opposed to that used in [34] implies that the expectations given by (3)–(10) are computed in a different way than in [34]. Here we first integrate with respect to t and then θ. It is easy to supply a source that would fit our requirements. This would be, for example, a radio source, a microwave, or a laser source, all with unstabilized phases. To realize this experiment, one must also use two additional coherent sources with stable known phases and with the same frequency as the nonstabilized source. If a data table is then built that keeps track of all the measured values on the detectors, we can compute the correlations and see a violation of Bell's inequalities.

There are several other remarks that we must add in order to clarify some points. First, when using classical fields the number of photons is overwhelmingly large. For that reason, we would not need to compute any photon count correlation. What we measure is intensity. On the other hand, Bell's inequalities are not enough to show that we do not have a joint probability distribution for classical fields, because they assume a continuous range of values. That is why we computed the correlation matrix, showing that for this case a joint probability distribution does indeed not exist.

Another point is that intensity of classical fields does not satisfy the basic assumption made by Bell, because it can take an infinite range of values; Bell considered spin measurements that can take only two possible values. To show that this does not present any constraint, we did an analysis of photon counts, which can only take, as in Bell's assumptions, two discrete values. However, to violate Bell's inequalities here, we must use an enhancement strategy. This strategy requires communication between the two different homodyne detectors, and it would not be non-local, as opposed to the true quantum example.

Finally, the last point. One can argue that if classical fields violate Bell's inequalities, then, since they are classical, Bell's theorem must be wrong, and we must show why it is wrong. First, we did not show that Bell's theorem is wrong; we just showed that a classical field approximates the quantum correlations and that Bell's inequalities are violated for them. Second, we emphasized that classical fields are continuous, and Bell's inequalities are derived from dichotomous random variables. Therefore, the conclusion that there does not exist a joint probability distribution for the intensities does not follow from violations of Bell's inequalities[3].

[3]In our arXiv:quant-ph/9606019 paper, we claimed that the correlation matrix for the intensities had negative and positive eigenvalues, which implied the non-existence of a joint probability distribution. This claim is incorrect, as our claim was based in a specific assumption for the hidden-variable model. However, as our example shows, a joint probability does exist,

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and can be constructed from the field itself. A quick examination reveals that the field-intensity model does not satisfy the assumptions of the hidden-variable model considered in our quant-ph paper.

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